VOLUME 65, NUMBER 26 PHYSICAL REVIEW LETTERS 24 DECEMBER 1990

Exact Solution for Quantum Davey-Stewartson I System

Gen-Di Pang
Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing, China

Fu-Cho Pu
Center of Theoretical Physics, CCAST (World Laboratory), Beijing, China
and Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing, China

Bao-Heng Zhao
Graduate School, Chinese Academy of Sciences, P.O. Box 3908, Beijing, China

(Received 15 August 1990)

The (2+1)-dimensional Davey-Stewartson I system is quantized, and the exact eigenvalues and eigenfunctions of the Hamiltonian are obtained. Thermodynamic quantities are also given. Although it is quantized by using the commutation relations for bosons, the energy spectrum and thus the thermodynamic quantities reveal the characteristics of a system with two species of independent fermions.

PACS numbers: 03.65.−w, 05.30.Jp

The integrable systems in a space of one spatial and one temporal dimension [(1+1)D] and the solvable planar statistical model (there is an intimate relationship between them) have aroused great interest in recent years. The exact solutions of the models are very helpful in understanding many interesting physical problems, e.g., the properties of magnetic alloys at low temperatures (the Kondo problem),[1] the critical behavior of two-dimensional systems,[2] etc. The study of the Yang-Baxter equation in the (1+1)D integrable systems and the two-dimensional solvable models of statistical mechanics[3,4] stimulated the development of many branches of mathematics, the most important examples being the theories of the quantum group and braid group.[4,5]

The next logical step in the research of integrable systems, of course, is to study the (2+1)D problems. Some classical nonlinear evolution equations in (2+1)D space were solved exactly in the last decade.[6] However, none of the quantum analogs was solved.

The Davey-Stewartson I (DSI) system[7] is one of the simplest of the known (2+1)D integrable systems. It can be considered as the nonlocal generalization of the (1+1)D nonlinear Schrödinger model or the δ-function gas model. The initial-boundary-value problem for the classical DSI equation can be solved exactly by using the inverse-scattering transform.[8] Efforts were made to study the quantized (2+1)D DSI system;[9] however, the energy spectrum and thermodynamics of it were still unknown, and the problem of the solution of the quantized DSI system remained open.

In this Letter we will solve the quantized DSI system to obtain the eigenvalues and eigenfunctions of the Hamiltonian, and the thermodynamic quantities. We will show that although it is quantized by the commutation relations for bosons, the energy spectrum and thus the thermodynamics reveal the characteristics of a system of two species of independent fermions, and they do not depend on the strength of the coupling between particles. For (1+1)D integrable systems, e.g., the nonlinear Schrödinger model, the bound states are caused by the attractive forces between particles, but for the DSI system, the bound states are given by suitable applied forces.

The DSI equation is

$$i\partial_t q = -\tfrac{1}{2}(\partial_x^2+\partial_y^2)q + iA_1 q - iA_2 q\,, \tag{1}$$

where

$$\begin{aligned}(\partial_x-\partial_y)A_1 &= -ic(\partial_x+\partial_y)(q^*q)\,,\\ (\partial_x+\partial_y)A_2 &= ic(\partial_x-\partial_y)(q^*q)\,,\end{aligned} \tag{2}$$

and c is the coupling constant.

For convenience, we introduce the coordinates $\xi = x + y$, $\eta = x - y$ in place of x,y. A_1 and A_2 depend on the choice of the boundary conditions for them. We consider the case of

$$\begin{aligned}A_1 &= -ic\,\partial_\xi\partial_\eta^{-1}(q^*q) - iu_1(\xi)\,,\\ A_2 &= ic\,\partial_\eta\partial_\xi^{-1}(q^*q) + iu_2(\eta)\,,\end{aligned} \tag{3}$$

where

$$\partial_\eta^{-1}(q^*q) = \tfrac{1}{2}\left[\int_{-\infty}^{\eta} d\eta' - \int_{\eta}^{\infty} d\eta'\right] q^*(\xi,\eta',t)\,q(\xi,\eta',t)\,,$$

and u_1 and u_2 are applied potentials. With these choices of A_1 and A_2, (1) can be written as

$$\begin{aligned}i\partial_t q = &-(\partial_\xi^2+\partial_\eta^2)q + c[\partial_\xi\partial_\eta^{-1}(q^*q) + \partial_\eta\partial_\xi^{-1}(q^*q)]q\\ &+[u_1(\xi)+u_2(\eta)]q\,.\end{aligned} \tag{4}$$

We quantize the system with the following commutation relations:

$$\begin{aligned}[q(\xi,\eta,t),q^*(\xi',\eta',t)] &= 2\delta(\xi-\xi')\delta(\eta-\eta')\,,\\ [q(\xi,\eta,t),q(\xi',\eta',t)] &= 0\end{aligned} \tag{5}$$

[note that $\delta(x-x')\delta(y-y')=2\delta(\xi-\xi')\delta(\eta-\eta')$]. It is readily seen that (4) can be written in the form

$$\partial_t q = i[H,q]\,, \tag{6}$$

where H is the Hamiltonian of the system,

$$H=\int d\xi\, d\eta\, \tfrac{1}{2}\{-q^*(\partial_\xi^2+\partial_\eta^2)q+\tfrac{1}{2}cq^*[(\partial_\xi\partial_\eta^{-1}+\partial_\eta\partial_\xi^{-1})(q^*q)]q+(u_1+u_2)q^*q\}\,. \tag{7}$$

We consider the eigenvalue problem for the system with N particles,

$$H|\psi\rangle = E|\psi\rangle\,, \tag{8}$$

where

$$|\psi\rangle=\int d\xi_1\, d\eta_1 \cdots d\xi_N\, d\eta_N\, \psi(\xi_1,\eta_1;\ldots;\xi_N,\eta_N)q^*(\xi_1,\eta_1)\cdots q^*(\xi_N,\eta_N)|0\rangle\,. \tag{9}$$

The time variable t is omitted. The vacuum state $|0\rangle$ has the property $q|0\rangle=0$. The N-particle wave function ψ is symmetric with respect to the permutation of coordinates (ξ_i,η_i), and it satisfies the Schrödinger equation

$$-\sum_{i=1}^{N}(\partial_{\xi_i}^2+\partial_{\eta_i}^2)\psi+c\sum_{i<j}[\delta'(\xi_{ij})\epsilon(\eta_{ij})+\delta'(\eta_{ij})\epsilon(\xi_{ij})]\psi+\sum_{i=1}^{N}[u_1(\xi_i)+u_2(\eta_i)]\psi=E\psi\,, \tag{10}$$

where $\xi_{ij}=\xi_i-\xi_j$, $\delta'(\xi_{ij})=\partial_{\xi_i}\delta(\xi_{ij})$, and $\epsilon(\xi_{ij})=1$ for $\xi_{ij}>0$, 0 for $\xi_{ij}=0$, and -1 for $\xi_{ij}<0$.

Suppose that $\{\mathcal{X}_l(\xi)\}$ and $\{\mathcal{Y}_k(\eta)\}$ are two complete sets of orthonormal eigenfunctions for the following 1D Schrödinger equations with appropriate boundary conditions, respectively:

$$[-\partial_\xi^2+u_1(\xi)]\mathcal{X}_l(\xi)=\alpha_l\mathcal{X}_l(\xi),\quad [-\partial_\eta^2+u_2(\eta)]\mathcal{Y}_k(\eta)=\beta_k\mathcal{Y}_k(\eta)\,, \tag{11}$$

where α and β are eigenvalues, the subscripts l (k) are quantum numbers, and different l (k) correspond to different eigenstates $\mathcal{X}_l$ $(\mathcal{Y}_k)$. Noting that $f(\xi)\delta'(\xi)=f(0)\delta'(\xi)-f'(0)\delta(\xi)$, the Schrödinger equation (10) for $N=2$ can be solved readily. The solution can be generalized to the case $N>2$. We have the following result:

$$\psi=\left(\prod_{i<j}[1-\tfrac{1}{4}c\epsilon(\xi_{ij})\epsilon(\eta_{ij})]\right)\left(\sum_P(-1)^P\prod_{i=1}^{N}\mathcal{X}_{l_{P_i}}(\xi_i)\right)\left(\sum_P(-1)^P\prod_{i=1}^{N}\mathcal{Y}_{k_{P_i}}(\eta_i)\right), \tag{12}$$

and

$$E=\sum_{i=1}^{N}(\alpha_{l_i}+\beta_{k_i})\,, \tag{13}$$

where P are $N!$ permutations. The validity of (12) and (13) can be checked directly by substituting them into (10).

Different sets $(l_1,l_2,\ldots,l_N)$ and $(k_1,k_2,\ldots,k_N)$, where $l_1<l_2<\cdots<l_N$ and $k_1<k_2<\cdots<k_N$, correspond to different eigenfunctions ψ. The totality of them form a complete set of the eigenfunctions for (10). This can be shown by a continuity argument[10] with respect to c as follows.

For $c=0$, irrespective of the permutation symmetry,

$$\psi=\prod_{i=1}^{N}\mathcal{X}_{l_i}(\xi_i)\prod_{i=1}^{N}\mathcal{Y}_{k_i}(\eta_i) \tag{14}$$

is the solution of (10), and the totality of ψ in (14) is a complete set of eigenfunctions for (10). If $c>0$ and $c\to 0^+$, using (10) it is easy to show that $\prod_{i=1}^{N}\mathcal{X}_{l_i}(\xi_i)$ and $\prod_{i=1}^{N}\mathcal{Y}_{k_i}(\eta_i)$ must be antisymmetrized, respectively. Thus in the limit $c\to 0^+$, the totality of

$$\psi=\left(\sum_P(-1)^P\prod_{i=1}^{N}\mathcal{X}_{l_{P_i}}(\xi_i)\right)\left(\sum_P(-1)^P\prod_{i=1}^{N}\mathcal{Y}_{k_{P_i}}(\eta_i)\right) \tag{15}$$

is a complete set of eigenfunctions for (10). Then by a continuity argument with respect to c, we see that for finite $c>0$ the totality of ψ in (12) is a complete set of the eigenfunctions for (10).

In (12), $l_i\neq l_j$ and $k_i\neq k_j$ if $i\neq j$, otherwise $\psi=0$. The energy spectrum and then the thermodynamic quantities reveal the characteristics of a system with two species of independent fermions. Neither of them depends on the value of the coupling constant c.

We give two examples.

(1) $u_1=u_2=0$. In this case, $\mathcal{X}_l$ and $\mathcal{Y}_k$ are plane waves. If we impose rigid boundary conditions on ψ, i.e., $\psi|_{\xi_i=0,L}=\psi|_{\eta_i=0,L}=0$ for all i, then

$$\psi=\left(\prod_{i<j}[1-\tfrac{1}{4}c\epsilon(\xi_{ij})\epsilon(\eta_{ij})]\right)\left(\sum_P(-1)^P\prod_{i=1}^{N}\sin(l_{Pi}\xi_i)\right)\left(\sum_P(-1)^P\prod_{i=1}^{N}\sin(k_{Pi}\eta_i)\right) \tag{16}$$

and

$$E=\sum_{i=1}^{N}(l_i^2+k_i^2)\,, \tag{17}$$

where $l = n\pi/L$ and $k = m\pi/L$ with $n,m = 1,2,\ldots$. The ground state has quantum numbers $\{n\} = \{1,2,\ldots,N\}$ and $\{m\} = \{1,2,\ldots,N\}$. At temperature β^{-1}, the average energy per particle and the pressure p of the system (with $L,N \to \infty$ and $D = N/L$ being finite) are

$$E_\beta/N = 2D^{-1}\int_0^\infty dk\,\rho(k)k^2\,,$$
$$p = (\pi\beta)^{-1}\int_0^\infty dk\ln\{1+\exp[-\beta(k^2-\mu)]\}\,, \tag{18}$$

where

$$\rho(k) = \pi^{-1}/\{1+\exp[\beta(k^2-\mu)]\}\,, \tag{19}$$

which satisfies $D = \int_0^\infty dk\,\rho(k)$, and μ is the chemical potential.

(2) $u_1(\xi) = K_1\xi^2/2$ and $u_2(\eta) = K_2\eta^2/2$ with the constants $K_n > 0$, $n = 1,2$. In this case, all the particles are bounded by the applied potentials u_1 and u_2 in finite range around $\xi = 0$, $\eta = 0$, and

$$\psi = \left(\prod_{i<j}[1-\tfrac{1}{4}c\epsilon(\eta_{ij})\epsilon(\xi_{ij})]\right)\left(\sum_P(-1)^P\prod_{i=1}^N H_{l_{P_i}}(\lambda_1\xi_i)\exp(-\tfrac{1}{2}\lambda_1^2\xi_i^2)\right)\left(\sum_P(-1)^P\prod_{i=1}^N H_{k_{P_i}}(\lambda_2\eta_i)\exp(-\tfrac{1}{2}\lambda_2^2\eta_i^2)\right) \tag{20}$$

and

$$E = \sum_{i=1}^N (2K_1)^{1/2}(l_i+\tfrac{1}{2}) + \sum_{i=1}^N (2K_2)^{1/2}(k_i+\tfrac{1}{2})\,, \tag{21}$$

where $\lambda_n = (K_n/2)^{1/4}$ $(n = 1,2)$, $l,k = 0,1,2,\ldots$, and the $H_l(\lambda\xi)$ in (20) are the Hermite polynomials. The ground state of the system corresponds to the quantum numbers $\{l\} = \{0,1,2,\ldots,N-1\}$ and $\{k\} = \{0,1,2,\ldots,N-1\}$. The average energy of the system at temperature β^{-1} is

$$E_\beta = \sum_{l=0}^\infty (2K_1)^{1/2}(l+\tfrac{1}{2})\rho_1(l) + \sum_{k=0}^\infty (2K_2)^{1/2}(k+\tfrac{1}{2})\rho_2(k)\,, \tag{22}$$

where

$$\rho_n(l) = 1/\{1+\exp[\beta((2K_n)^{1/2}(l+\tfrac{1}{2})-\mu_n)]\}\,,\quad n = 1,2\,, \tag{23}$$

which satisfy $N = \sum_{l=0}^\infty \rho_1(l) = \sum_{k=0}^\infty \rho_2(k)$.

Calculations for other potentials u_1 and u_2 are straightforward, provided that the eigenvalue problem (11) has been solved. The Hamiltonian (7) can also be generalized to higher dimensions and the details will be published elsewhere.

---

[1] N. Andrei, Phys. Rev. Lett. **45**, 379 (1980); P. B. Wiegmann, Pis'ma Zh. Eksp. Teor. Fiz. **31**, 392 (1980) [JETP Lett. **31**, 364 (1980)]; A. M. Tsvelick and P. B. Wiegmann, Adv. Phys. **32**, 453 (1983); N. Andrei, K. Furuya, and J. H. Lowenstein, Rev. Mod. Phys. **55**, 331 (1983).

[2] For reviews, see R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982); *Conformal Invariance and Applications to Statistical Mechanics*, edited by C. Itzykson, H. Saleur, and J. B. Zuber (World Scientific, Singapore, 1988).

[3] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); R. J. Baxter, Ann. Phys. (N.Y.) **70**, 193 (1972).

[4] *Yang-Baxter Equation in Integrable Systems*, edited by M. Jimbo (World Scientific, Singapore, 1989).

[5] V. G. Drinfeld, Sov. Math. Dokl. **32**, 254 (1985); M. Jimbo, Lett. Math. Phys. **10**, 63 (1985); Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. **56**, 839 (1987); **56**, 3039 (1987); *Braid Group, Knot Theory and Statistical Mechanics*, edited by C. N. Yang and M. L. Ge (World Scientific, Singapore, 1989).

[6] For reviews, see A. S. Fokas and M. J. Ablowitz, in *Nonlinear Phenomena*, edited by K. B. Wolf, Lecture Notes in Physics Vol. 189 (Springer-Verlag, Berlin, 1983); M. J. Ablowitz and A. I. Nachman, Physica (Amsterdam) **18D**, 223 (1986).

[7] A. Davey and K. Stewartson, Proc. Roy. Soc. London A **338**, 101 (1974).

[8] A. S. Fokas, Phys. Rev. Lett. **51**, 3 (1983); A. S. Fokas and M. J. Ablowitz, Phys. Rev. Lett. **51**, 7 (1983); J. Math. Phys. **25**, 2494 (1984); A. S. Fokas and P. M. Santini, Phys. Rev. Lett. **63**, 1329 (1989); P. M. Santini, Physica (Amsterdam) **41D**, 26 (1990).

[9] C. L. Schultz, M. J. Ablowitz, and D. Bar Yaacov, Phys. Rev. Lett. **59**, 2825 (1987); P. P. Kulish and V. D. Lipovsky, Phys. Lett. A **127**, 413 (1988).

[10] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).