Space-Filling Bearings

H. J. Herrmann, G. Mantica, and D. Bessis
Service de Physique Théorique de Saclay, 91191 Gif-sur-Yvette, France
(Received 2 August 1990)

A model for dense packings of disks rolling on each other is presented. This model might have application for turbulence, tectonic motion, and mechanical gearworks. A full classification of solutions with fourfold loops is given. The fractal dimensions are calculated and compare favorably with Kolmogoroff scaling of turbulence.

PACS numbers: 03.20.+i, 02.20.+b, 03.40.Gc, 91.45.Dh

Is it possible to tile an infinite strip with wheels rolling on each other such that the entire area is covered with wheels? This question can arise in various contexts. One could imagine the wheels to be circular eddies on the surface of an incompressible fluid and then ask if the fluid motion can be totally decomposed into stable eddies. Or, one could think of mechanical roller bearings between two moving surfaces, like two tectonic plates, and then ask if one can completely fill the space between the rolling cylinders with other rolling cylinders such that no cylinder exerts any frictional work on another one. The question we are asking is, in fact, geometrical.

The original motivation for studying this problem was the enigmatic observation that over very extended areas, called "seismic gaps,"[1] two tectonic plates can creep on each other without producing either earthquakes or the amount of heat expected from usual friction forces. As a possible mechanism to explain this behavior one could think that the material between the plates, which is called "gouge," organizes itself in such a way that it acts like a bearing. Since in a bearing one has rolling friction but no gliding friction, this could explain the lack of measurable heat production. Under the pressure of kilobars the space between the rolling cylinders must be filled with rolling matter such that the motion of the main cylinders is not hindered. So, the main cylinders should roll on secondary cylinders which themselves roll on successive generations of smaller and smaller cylinders. Many physical effects taking place in tectonic plates have been neglected in the above argument, yet purely geometrical reasons lead us to believe that if such bearings were to exist, they could be constructed only iteratively, and would therefore be self-similar. (We will not discriminate between self-similar and self-inverse.[2]) Several seismic measurements have in fact indicated self-similar or turbulent motion within the gouge.[3]

In this Letter, we show that it is indeed possible to have space-filling bearings, such as those shown in Fig. 1. We will discuss the full set of solutions, their construction algorithms, and physical properties, which show interesting similarities with turbulence in the inertial regime.

Tiling the space with circles by putting iteratively in each hole between three circles the circle that tangentially touches all three is an old problem known as "Apollonian packing." It dates back to 200 B.C. and much work has been done since.[2] The Apollonian packing defines a fractal whose dimension is about 1.3.[4] Here we need more general packings, which can be obtained—as the Apollonian packing[2]—using iteratively circle-conserving mappings (Möbius transformations[5]). In the complex z-plane, Möbius transformations are given by $z'=(az+b)/(cz+d)$. The constants a, b, c, and d will be written as the entries of a 2×2 matrix M, which fulfills $\mathrm{Det}(M)=ad-bc=1$. Möbius transformations can be decomposed into translations, rotations, reflections, and inversions. (An inversion about a circle Γ is a Möbius transformation that maps conformally the interior of Γ to the exterior, and vice versa.) Only inversions, however, change the size of the circles and will therefore constitute the central element in the iteration of contracting mappings that we need.

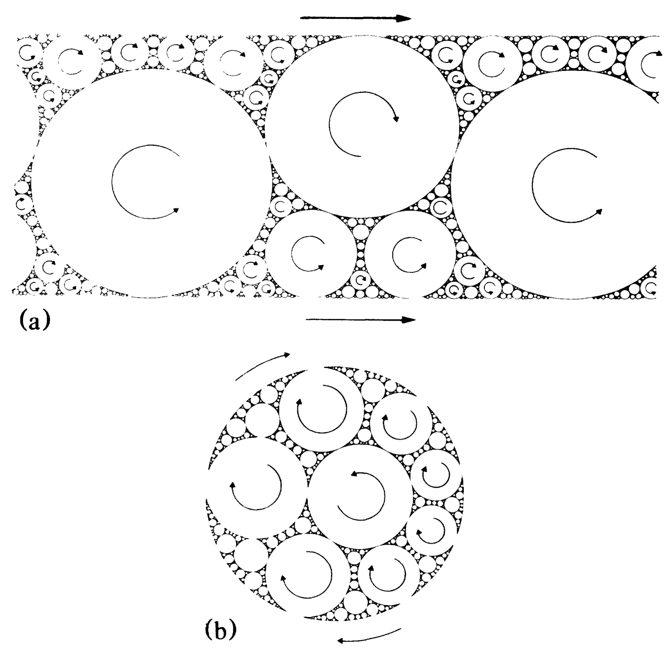

FIG. 1. Rolling, space-filling bearings (a) $n=4$, $m=2$ of the first family of fourfold loops in strip geometry; (b) $n=m=4$ of the first family in circular geometry.

The main problem one has to solve in order to construct space-filling packings is that of filling the wedge between two tangentially touching circles. This can be achieved by mapping an infinite strip into the wedge by an appropriate inversion. The strip itself can be constructed by periodically repeating a given pattern of circles.

The other important condition we must fulfill is the slipless rotation of each disk on its tangential neighbors. Disks can rotate either clockwise or counterclockwise. A clockwise-rotating disk can only touch counterclockwise-rotating ones, and vice versa. Consequently, any loop of touching disks in the packing must have an even number of disks. Suppose one constructs one of these loops by starting with one disk and adding one by one the disks of the loop. If the first disk rotates with tangential velocity v, its touching neighbor will have the same tangential velocity; therefore, when one closes the loop, the last disk will not encounter any slip at the two contact points. The problem can therefore be shown to reduce to the construction of packings with only *even* loops.

It is possible to give a general algebraic approach to this problem, using Coxeter algebras.[6] Here we shall give a geometrical construction of packings with fourfold loops following the line developed by one of us.[7] The basic construction is shown in Fig. 2(a), for a strip of unit width. The largest circle (B) is set tangent to the bottom line of the strip $\hat{C}_0$ at the point I_B, and the second largest circle (A) lies tangent to the top of the strip (C_0) at the point I_A. Since A and B are also tangent, the problem is parametrized by R_A and R_B, the radii of A and B. Let Δ_A and Δ_B be lines orthogonal to C_0 at I_A and $\hat{C}_0$ at I_B, respectively. We choose $\hat{C}_0$ and Δ_B as real and imaginary axes of the complex z plane. We use three Möbius transformations in SL(2,C) to generate the circle-filling figure by acting on the original circles A and B, and then iteratively on their images. These generators are as follows.

(1) A translation $\mathcal{T}$ parallel to the strip: $\mathcal{T}(z)=z+2a$. To fix a, we use the fact that A is tangent to $\mathcal{T}(B)$ in Fig. 2(a), i.e.,

$$2(R_A+R_B)=1+a^2. \qquad (1)$$

(2) An inversion $\mathcal{J}_B$ which preserves $\hat{C}_0$, Δ_B, and A. This is an inversion with respect to a circle Γ_B of center I_B and radius r_B. Requiring C_0 to be mapped into B gives $r_B^2=2R_B$. To avoid antianalytic transformations, we multiply $\mathcal{J}_B$ on the left by $\mathcal{R}$, a symmetry transformation with respect to the line $x=a$, and get a Möbius map whose matrix representation is

$$\mathcal{R}\mathcal{J}_B=\begin{pmatrix} 2a/r_B & -r_B \\ 1/r_B & 0 \end{pmatrix}.$$

(3) An inversion $\mathcal{J}_A$ which preserves C_0, Δ_A, and B. This is an inversion with respect to a circle Γ_A of radius r_A. To map $\hat{C}_0$ into A we need $r_A^2=2R_A$. Since $(\mathcal{R}\mathcal{J}_A)^2$ is the identity, and as such cannot produce any new circles, we take as the third generator

$$\mathcal{R}\mathcal{J}_A\mathcal{T}=\begin{pmatrix} (a+i)/r_A & (a^2+1-r_A^2)/r_A \\ 1/r_A & (a-i)/r_A \end{pmatrix}.$$

All the matrices of our generators, $\mathcal{T}$, $\mathcal{R}\mathcal{J}_B$, and $\mathcal{R}\mathcal{J}_A\mathcal{T}$, have unit determinants. In order to avoid overlapping of the circles these can only be translations or discrete rotations of $2\pi/(n+3)$, $n\geq 0$ (Ref. 6) (i.e., their eigenvalues must be ± 1 or $e^{\pm 2i\pi/(n+3)}$). This implies that

$$\mathrm{Tr}\{\mathcal{R}\mathcal{J}_B\}=2a/r_B=2\cos[\pi/(m+3)],$$

and

$$\mathrm{Tr}\{\mathcal{R}\mathcal{J}_A\mathcal{T}\}=2a/r_A=2\cos[\pi/(n+3)].$$

Defining $z_n=\cos^{-2}[\pi/(n+3)]$, the previous relations, together with Eq. (1), provide the solution of the filling problem:

$$a^{-2}=z_n+z_m-1,\quad r_A^2=a^2z_n,\quad r_B^2=a^2z_m. \qquad (2)$$

This defines a first family of space-filling circles depending on two discrete indices m and n, which can take all integer values from zero to infinity. The necessary condition (2) can be shown to be sufficient through explicit construction: A few members of this family are illustrated in Fig. 3.

Beyond this case, only a second family exists having basic fourfold loops, and is obtained by setting the distance between Δ_A and Δ_B equal to zero [see Fig. 2(b)]. In this case, the group generators $\mathcal{T}$ and $\mathcal{R}\mathcal{J}_B$ remain

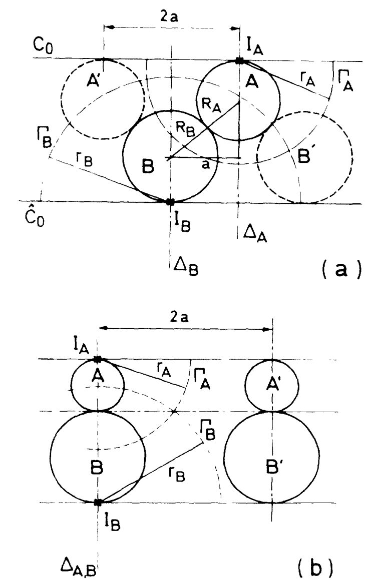

FIG. 2. Schematic position of the largest circles in (a) the first family and (b) the second family.

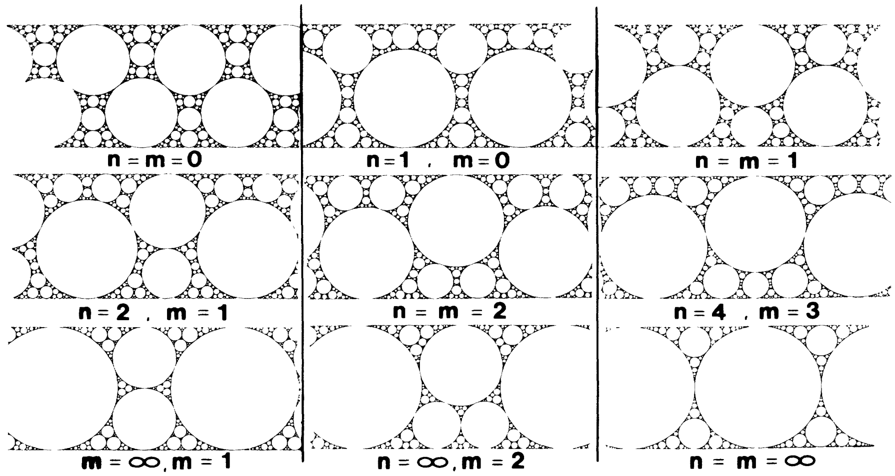

FIG. 3. Nine different combinations of n and m of the first family.

the same; however, $\mathcal{R}\mathcal{I}_A\mathcal{T}$ is replaced by

$$\mathcal{R}\mathcal{I}_A=\begin{pmatrix}(2a+i)/r_A & (2ia+1-r_A^2)/r_A\\ 1/r_A & -i/r_A\end{pmatrix},\quad r_A^2=2R_A\,.$$

Taking the trace, we get the same relations as before; however, the geometrical constraint (1) is now replaced by $2(R_A+R_B)=1$, and (2) is changed into $a^{-2}=z_n+z_m$, $r_A^2=a^2z_n$, $r_B^2=a^2z_m$. By direct construction, one can also show that this second family generates a complete filling of the strip.

We have now found all the regular packings constructed with fourfold loops. It is, however, also possible to mix these solutions: Since the only shape that two different packings have in common is the circle, the mixing is best done by conformally mapping the strip into a circular geometry [see Fig. 1(b)] by inverting the strip about the circle B. One can now fill the interiors of circles of a given regular packing by randomly chosen solutions (n,m) in the circular geometry. This is the most general packing of fourfold loops one can construct.

For solutions with higher, even coordination of loops, we also expect to find only solutions labeled by integers, because the origin of the discreteness of the solutions [Eq. (2)] is independent of the coordination condition. It would be useful to find the full classification for arbitrary coordination, using, for instance, Ref. 6.

The packings that we have constructed are evidently fractal. One way to define their fractal dimension is by introducing a cutoff length ϵ and considering only circles with radii larger than ϵ. One can calculate numerically the number $N(\epsilon)$ of such circles, the sum $s(\epsilon)$ of their perimeters ("surface"), and the "porosity" $p(\epsilon)$ (i.e., the area not covered by circles), all per unit area. These quantities can be related to the distribution $n(r)$ of circles of radius r per unit area through

$$N(\epsilon)=\int_\epsilon^\infty n(r)\,dr,\quad s(\epsilon)=2\pi\int_\epsilon^\infty rn(r)\,dr\,,$$
$$p(\epsilon)=1-\pi\int_\epsilon^\infty r^2n(r)\,dr\,. \tag{3}$$

If $n(r)$ can be described by a simple power law $n(r)\sim r^{-\tilde\tau}$, then one finds

$$N(\epsilon)\sim\epsilon^{-d_f},\quad s(\epsilon)\sim\epsilon^{1-d_f}\,,$$
$$p(\epsilon)\sim\epsilon^{2-d_f},\quad d_f=\tilde\tau-1\,,$$

where d_f is the fractal dimension.[2]

In Fig. 4 we show N, s, and p plotted double logarithmically against the cutoff for $n=m=0$ and for $n=m=\infty$ in the first family. The fact that the porosity goes to zero with ϵ is a numerical verification that the packings are space filling. The fractal dimension one obtains from the porosity for the Apollonian packing agrees well with Boyd's value[4] and seems to be the same for the whole "Apollonian family" ($n=\infty$ in the first family).

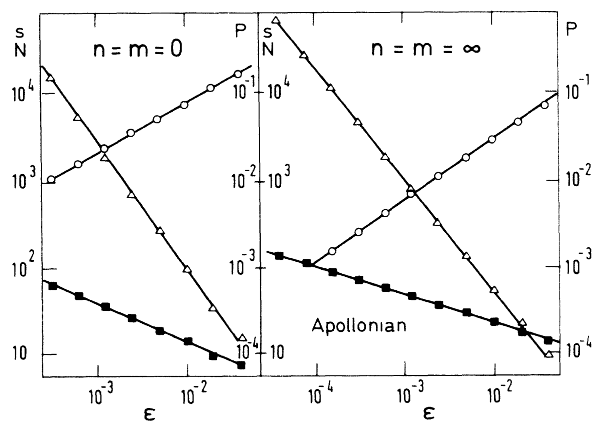

FIG. 4. A log-log plot of the number of circles N, the surface s, and the porosity p as function of the cutoff ϵ for the first family. In the $n=m=0$ case, the fractal dimensions d_f obtained from N, s, and p are 1.47, 1.45, and 1.42.

For the first family the fractal dimension obtained from the porosity continuously increases with decreasing n and m and is 1.42 for $n=m=0$. For the second family one finds 1.52 for $n=m=0$.

Since all disks rotate with the same tangential velocity v, the kinetic energy of disks of radius r is $E(r) \propto v^2 r^2 \times n(r)$, and the energy spectrum (as a function of the wave vector $k=r^{-1}$) is

$$E(k)dk \sim k^{\tilde{\tau}-4}dk = k^{d_f-3}dk \, .$$

(It is interesting to note that the total energy per unit surface is constant for any space-filling system: It is infinitely degenerate.) We found values of d_f ranging from 1.30 to 1.52, consistent with the Kolmogoroff scaling of the energy spectrum of homogeneous fully developed turbulence:[8] $E(k)dk \sim k^{-5/3}dk$. This interesting coincidence might have a more profound meaning and might give a geometrical interpretation of turbulence as a picture of energy transfer to smaller and smaller eddies in the inertial regime.

Our model does not produce any dissipation, as turbulence in the inertial regime. Furthermore, its construction shares similar characteristics with the β model of turbulence.[9] A more direct realization of our rolling packings is presently realized experimentally by rotating a fluid between two or more (eccentric) cylinders as shown in Fig. 1(b). If the position and size of the inner cylinder corresponds to one of our solutions, and one compensates for the viscosity of the fluid, one might expect the fluid to convect along the smaller intermediate circles in Fig. 1(b).

We thank L. Knopoff and K. R. Sreenivasan for useful discussions.

[1]W. McCann, S. Nishenko, L. Sykes, and J. Krause, Pure Appl. Geophys. **117**, 1082 (1979); C. Lomnitz, Bull. Seismol. Soc. Am. **72**, 1441 (1982).

[2]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[3]C. Sammis, G. King, and R. Biegel, Pure Appl. Geophys. **125**, 777 (1987).

[4]D. W. Boyd, Mathematica **20**, 170 (1973); see also Ref. 2.

[5]K. J. Falconer, *The Geometry of Fractal Sets* (Cambridge Univ. Press, New York, 1985); L. R. Ford, *Automorphic Functions* (Chelsea, New York, 1929).

[6]G. Maxwell, J. Algebra **79**, 78–97 (1982).

[7]D. Bessis and S. Demko, Service de Physique Théorique, CEA Saclay, Report No. SphT 90/018 (to be published).

[8]G. K. Batchelor, *Theory of Homogeneous Turbulence* (Cambridge Univ. Press, New York, 1982).

[9]U. Frisch, P. L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719–736 (1978).