Nonquasilinear Diffusion Far from the Chaotic Threshold

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The diffusion coefficient for particles in a field of randomly phased waves was calculated numerically. Spectra broad in wave number and broad in frequency exhibit identical behavior for the diffusion coefficient as a function of the overlap parameter ε. The diffusion coefficient exceeds the quasilinear value by a factor of about 2.5 at ε near 18. Differences greater than 25% between the two values exist far above the chaotic threshold ($\varepsilon \approx 60$). A mapping-based model with a small number of random phases qualitatively reproduces the results of the simulations.

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The theory of diffusion[1] of particles in a turbulent field has attracted much attention due to its relevance to weak plasma turbulence and anomalous transport.[2] Early theories relied on the randomness of the wave phases, thus indicating that diffusion is connected with *extrinsic* randomness. But more recently it was realized[3] that diffusion can be an *intrinsic* property of a deterministic system, whose presence depends on the parameters (e.g., Chirikov's overlap parameter) describing the dynamics. For example, the diffusion coefficient for the standard map,[3] a dynamical system of regularly phased waves, was found to rise from zero at the threshold for global chaos. The ratio of the diffusion coefficient to the quasilinear value then oscillates[4] as the kick amplitude is increased.

The enhancements of the diffusion coefficient in the standard map are often connected with accelerator modes,[3] stable regions of phase space where the momentum changes linearly with time. Such stable regions are not likely to exist for the randomly phased field. (Indeed, such regions are not observed in our numerical calculations.) Accordingly, the diffusion coefficient might be expected to rise monotonically from zero to the quasilinear value with increasing force amplitude. This is expected from Dupree's turbulence theory,[5] which indicates that the intrinsic chaos cuts off the correlation function, thereby decreasing the integral giving the diffusion coefficient. Moreover, monotonic behavior of the diffusion coefficient was apparently observed in previous numerical simulations.[6]

Surprisingly, we find that the diffusion coefficient D/D_{QL} relative to its quasilinear value rises to a maximum of about 2.3 at an overlap parameter of $\varepsilon = 18$ and then *decays* to its asymptotic value. Moreover, significant differences (of the order of 25%) remain far above the chaos threshold (by a factor of 60). Assuming the interaction is local in velocity (see below), we show that these results are quite general. They are valid for spectra with large frequency spread or large wave-number spread. We are able to reproduce the diffusion coefficient qualitatively with the correlation-function formalism[7] modified to allow random phases.

Our results are connected to those of recent studies[8] in the regime of "turbulent trapping." Here the field can be decomposed into a sum of wave packets with envelopes barely overlapping spatially and resonance widths barely overlapping in velocity. In this regime a diffusion-coefficient enhancement by a factor of approximately 2 has been found. We study a regime similar in that our resonance widths are barely overlapping, but we have plane waves. Furthermore, our case is not self-consistent and, thus, may not have certain self-consistency-induced wave correlations. However, our case may be viewed as the instantaneous state of a slow, self-consistent evolution with randomly phased waves.

The Hamiltonian for a particle moving in a spectrum of randomly phased waves with varying wave numbers and frequencies is

$$H = \tfrac{1}{2} p^2 + \varepsilon \sum_{n=0}^{N} b_n \cos(k_n q - \omega_n t + \phi_n) , \qquad (1)$$

where the ϕ_n's are chosen randomly. With slowly varying wave parameters, the resonant velocities $v_{\phi n} \equiv \omega_n / k_n$ are spaced by $\Delta v_{\phi n} = k_n^{-1} \partial \omega_n / \partial n - (\omega_n / k_n^2) \partial k_n / \partial n$ near the nth wave. A modified overlap criterion,[3] that the resonance widths of the primary islands be $2/\pi$ of the resonant-velocity separations, predicts the disappearance of the last Kolmogorov-Arnol'd-Moser curves of the original topology near the nth resonance at the amplitude value, $\varepsilon b_n = \Delta v_{\phi n}^2 / 4\pi^2$.

The linear-orbit approximation gives the diffusion coefficient as a sum of δ functions at the resonant velocities. Smoothing over these δ functions gives the quasilinear diffusion coefficient $D_{\mathrm{QL}} = \pi \varepsilon^2 k_n b_n^2 / 2 \Delta v_{\phi n}$, which is valid[9] in the limit of large ε. Our goal is to determine how the true diffusion coefficient D in this system differs from the quasilinear value D_{QL} for intermediate values of ε: above the chaotic threshold but before the quasilinear limit is reached.

We assume that the wave-particle interaction obeys *locality*. By *locality* we mean that D depends on only those parameters, k_n, b_n, ω_n, describing the waves nearly resonant with the considered velocity. To use locality we first make a Galilean transformation of velocity ω_{n_0} / k_{n_0}

to the frame of the most nearly resonant wave. This gives the Hamiltonian

$$H=\tfrac{1}{2}p^2+\varepsilon\sum b_n\cos\{k_n[q-(v_{\phi n_0}-v_{\phi n})t]+\phi_n\}\,.$$

Second, we assume that the b_n's and k_n's are sufficiently slowly varying that we may set them to b_{n_0} and k_{n_0}. Similarly, we make the replacement $v_{\phi n_0}-v_{\phi n}\rightarrow(n_0-n)\Delta v_{n_0}$. Next, we rescale the system so that k_{n_0} and Δv_{n_0} become unity, and we define b_{n_0} so that the modified overlap criterion mentioned above occurs at $\varepsilon=1$. Finally, we reindex to put the index of the most nearly resonant wave at zero. Thus, locality allows us to replace the Hamiltonian (1) by

$$H(p,q,t)=\tfrac{1}{2}p^2+\frac{\varepsilon}{4\pi^2}\sum_{m=-N_1}^{N_2}\cos(q-mt+\phi_m)\,, \qquad (2)$$

in which $N_1=n_0$ and $N_2=N-n_0$ are large. For infinite N_1 and N_2, this Hamiltonian would be that of the standard map,[3] were all the phases zero.

The diffusion coefficient for this system was measured by initiating an ensemble of 400 particles uniform in position with unique initial velocity and integrating with a leapfrog integrator using a time step one-tenth of the shortest wave period. (After making convergence tests, we chose $N_1=N_2=300$.) Half $d\langle\Delta p^2\rangle/2dt$ of the large-t slope of the spread of this ensemble gives the diffusion coefficient. Such measurements must be made at an intermediate time, such that the ensemble is chaotically mixed, yet the particles have not encountered the velocity limits of the chaotic region.

The squares in Fig. 1 are the measured ratio of the diffusion coefficient to the quasilinear value. This ratio is seen to have a maximum of about 2.3 near $\varepsilon=18$. Diffusion faster than quasilinear was observed in the standard map,[4] but there it was related to accelerator modes, whose existence relies on the special set of phases (all zero) for the standard map. In the random-phase case coherent structures are not expected (nor observed in our simulations), yet the diffusion coefficient is significantly above quasilinear. Additionally, for the standard map the ratio $D/D_{\rm QL}$ is oscillatory with local minima well below unity. For the random-phase case this ratio never falls below the unity after rising above it.

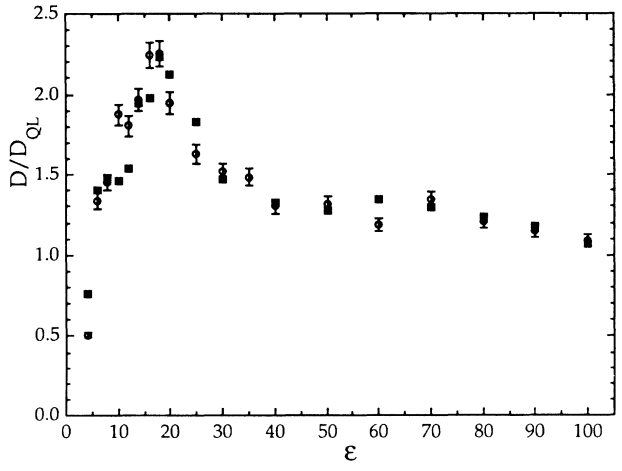

FIG. 1. Ratio of the diffusion coefficient to the quasilinear value shown as squares for the Hamiltonian (2) and as circles for the broad-wave-number case (1).

We tested the independence of our results on the initial conditions and the phase choice. We selected the initial coordinates evenly spaced in the interval $[0,2\pi]$ and also by a random-number generator with various seeds. We selected various values for the initial velocity. We used different seeds for the random generation of the wave phases. To within statistical noise, the diffusion coefficient was unaffected.

To check locality, we studied a Hamiltonian with a broad-wave-number spectrum. To facilitate the measurements, we chose parameters to have uniform overlap and quasilinear diffusion coefficient. This implies $k_n=(k_0^{-2/3}-Bn)^{-3/2}$ and $b_n=(3\omega B/4\pi)^2k_n^{-2/3}$, with $B=(k_0^{-2/3}-k_N^{-2/3})/N$. We chose $k_0=2$ and $k_N=\frac{1}{2}$, with $N=200$. For each value of ε, 5000 particles were integrated. The results are shown by the circles in Fig. 1. The error bars were determined from the variations among subsets of the full ensemble. (The error bars for the squares should be roughly 3.5 times larger as that ensemble was smaller by a factor of 12.) That the two systems give the same result verifies locality. The diffusion coefficient depends on neither the type of spectrum nor its large-scale properties.

To analytically compute the diffusion coefficient we consider the Hamiltonian

$$H_M(p,q,t)=\frac{p^2}{2}+\frac{\varepsilon}{2\pi}\sum_{m=0}^{M-1}a_m\cos(q-\psi_m)\times\delta_{2\pi}(t-2\pi m/M)\,, \qquad (3)$$

a successive application of M phase-shifted standard maps of varying amplitude. In Eq. (3) is $\delta_{2\pi}$, the periodic δ function of period 2π. The mapping for this Hamiltonian is

$$p_n=p_{n-1}+(\varepsilon/2\pi)a_{n-1}\sin(q_{n-1}-\psi_n)$$

and $q_n=q_{n-1}+(2\pi/M)p_n$, where $a_{n+M}\equiv a_n$, $\psi_{n+M}\equiv\psi_n$, and q_n and p_n are the values of the coordinates just after the nth kick. The δ functions cause the Hamiltonian (3) to have an infinite set of resonances. Thus, it cannot be made to correspond exactly to the Hamiltonian (2). However, for the choice

$$a_m e^{-i\psi_m}=\frac{1}{M}\sum_{n=n'}^{n'+M-1}e^{-i(\phi_n+2\pi mn/M)}\,, \qquad (4)$$

the resonances of the Hamiltonian (3) all have the same amplitude, $1/4\pi^2$, as those of the Hamiltonian (2). In addition, the phases ϕ_n for $n'\le n\le n'+M-1$ are duplicated. Thus, with locality and sufficiently large M the

two Hamiltonians (2) and (3) have the same diffusion coefficient.

The diffusion coefficient for the Hamiltonian (3) is obtained from a modified characteristic-function formalism.[7] The diffusion coefficient obeys

$$D/D_{\mathrm{QL}}-1\equiv\sum_{n=1}^{\infty}\Delta_n\,, \tag{5}$$

where

$$\Delta_n=\frac{4}{M\varepsilon^2}\sum_{j=0}^{M-1}C_{j,n} \tag{6}$$

is the normalized average over the M steps of the map (3) of the phase-space-averaged impulse correlation function $C_{j,n}\equiv\langle(p_{j+1}-p_j)(p_{j+n+1}-p_{j+n})\rangle$. Ergodicity implies that $C_{j,n}$ is periodic in the index j with period M. The first correction Δ_1 vanishes identically. For the two-step mapping, the second correction is

$$\Delta_2=-2\cos^2(\Delta\phi/2)J_2(\varepsilon\sin(\Delta\phi/2)/2) -2\sin^2(\Delta\phi/2)J_2(\varepsilon\cos(\Delta\phi/2)/2)\,, \tag{7}$$

where $\Delta\phi\equiv\phi_1-\phi_0$ is the phase difference of the two resonances.

The random-phase diffusion coefficient is obtained by invoking the statistical hypothesis that in the limit of a large number of waves, the result for almost any choice of phases is very close to the result averaged over all the phases. This allows us to average Eq. (5) over the phases of the system. For example, averaging the second correction (7) of the two-step mapping yields $\langle\Delta_2\rangle=J_1^2(\varepsilon/4)/\pi-2J_2(\varepsilon/4)J_0(\varepsilon/4)/\pi$.

Figure 2 shows the two-step mapping results for the sum of the phase-averaged corrections containing at most two Bessel functions. These include correlations through $\langle\Delta_3\rangle$ and the principal terms[7] of $\langle\Delta_4\rangle$. These results show that the two-step mapping with a single-phase averaging is able to reproduce qualitatively the dependence of the diffusion coefficient on ε. Like the numerical results, the analytic diffusion coefficient has a large peak at moderate ε, and it never falls below the quasilinear value after rising above it. Unfortunately, the magnitude of the analytically calculated peak is too small by about 20%, and its location is at a value of ε too small by 40%.

The qualitative correctness of the two-step mapping calculation through $\langle\Delta_4\rangle$ indicates that correlations must be calculated through two periods ($\Delta t=4\pi$) of the Hamiltonian (2) to obtain the diffusion enhancement at $\varepsilon\approx 20$, where it is greatest. In physical units, this corresponds to a time $4\pi/k\Delta v$, since we had chosen the wave number k and the phase-velocity separation Δv to be unity in the Hamiltonian (2). Thus, for $\varepsilon\approx 20$ the nonlinear correlation time τ_{nc} is of the order of a bounce period, $\tau_{\mathrm{nc}}\approx 10/k(\varepsilon b_n)^{1/2}=10/\omega_{Bn}$, where ω_{Bn} is the bounce frequency for the nth mode. (This time is much longer than the linear-orbit correlation time, which is proportional to the inverse of the spectral width and, thus, vanishes in the limit of infinite spectral width.) Unfortunately, we cannot generalize this argument to arbitrary ε to say whether the nonlinear correlation time scales with ε as the individual bounce time ($\propto\varepsilon^{-1/2}$), the time to diffuse across a fixed number of resonances ($\sim 1/D\propto\varepsilon^{-2}$), or the resonance broadening time [$\sim 1/(k^2D)^{1/3}\propto\varepsilon^{-2/3}$], as all of these times are of the same order at $\varepsilon\approx 20$.

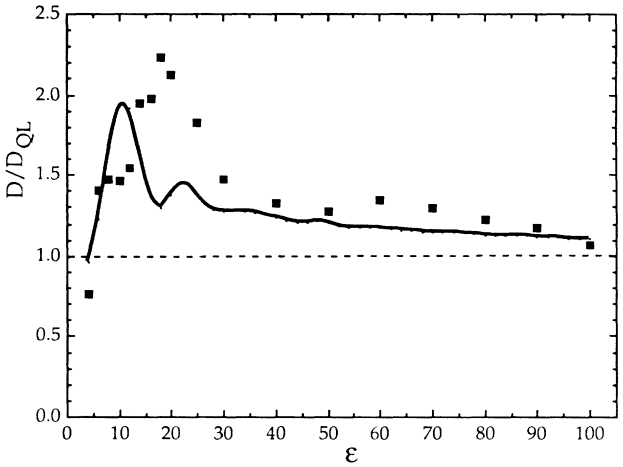

FIG. 2. Ratio of the diffusion coefficient to the quasilinear value shown as squares for the Hamiltonian (2) and as a solid curve for the randomly phased, two-step mapping (3).

Correlations over two periods in the three-step mapping involve terms up to Δ_6, which we have not calculated. Keeping the corrections only through Δ_4 shows a better fit to the location of the maximum in D/D_{QL} (it is shifted to 15), but produces a maximum of only 1.5. These results and those of the two-step mapping indicate that one need average over only very few phases (2–3) in order to see the behavior of a randomly phased system.

Lack of sufficient measuring time appears to have prevented the observation of nonquasilinear diffusion in previous work.[6,10] Linearized-orbit theory dictates quasilinear spreading for velocities away from the edges of the phase-velocity spectrum and for times between the spectral correlation time and the time at which orbit linearity (constant p) is no longer valid. Our numerical results show that to observe the deviations from quasilinear theory for $\varepsilon\approx 20$, the initial distribution must be followed until it has spread over roughly the nearest four resonances. Hence, it could have been that the simulation time of Ref. 6 was too short. In Ref. 10, Eq. (34), it is explicitly stated that the early-time diffusion coefficient is measured. Interestingly, the diffusion was observed to deviate from quasilinear in Ref. 10 [cf. Fig. 5(b)] at the time the distribution was spread over two resonances. However, in this work there were only thirteen resonances, and the diffusion coefficient varied by an order of magnitude over the corresponding range of velocities, so the deviation was attributed to inhomo-

geneity and boundary effects.

In conclusion, we have shown that the diffusion coefficient for a particle moving in a system of randomly phased waves is a function of only the local overlap parameter. This diffusion coefficient may exceed the quasilinear value by a factor of 2 in a regime reasonably far (by a factor of 20) from the chaotic threshold. Our mapping-based analysis indicates that the randomness of the phases can be modeled with only a very small number of random phases.

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