Extension of the Kasteleyn-Fortuin Formulas to Directed Percolation

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It is shown that the pair connectedness for percolation on an arbitrary directed graph may be obtained by setting $\lambda=1$ in the transmissivity function for a λ-state chiral Potts model, the odd-flow model, and that the pair connectedness of the dual percolation model is expressible in terms of the correlation function of a second chiral Potts model, the odd potential-difference model. There is an important distinction between odd and even λ and only those models with odd λ exhibit chirality. The duality relation between the transmissivity and correlation functions leads to a duality relation for pair connectedness of directed percolation theory which extends the result of Dhar, Barna, and Phani for percolation probabilities.

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Kasteleyn and Fortuin[1] related bond percolation theory on an arbitrary undirected graph G to the statistical mechanics of interacting spins via the λ-state Potts model.[2] In particular, they showed that setting $\lambda=1$ in the Potts model correlation function gave the pair connectedness function of percolation theory. This relation has been fundamental in allowing the cross fertilization of ideas between the two subjects and placed percolation models in the general context of critical phenomena theory. One of the more important consequences of the result was that it enabled the exact values of the critical exponents associated with the pair connectedness on the square lattice to be obtained by substituting $\lambda=1$ in the formula for the Potts model exponents.[3]

Numerical values of the corresponding critical exponents for directed percolation have been accurately calculated[4] and are clearly distinct from those of the undirected problem, yet no exact values have been obtained. There has also been considerable recent activity in obtaining exact results for chiral Potts models which has led to new exponent values.[5] The connection developed here between such models and directed percolation is a natural extension of the Kasteleyn-Fortuin result for undirected percolation. It is hoped that this link will enable the exact exponents associated with the pair connectedness for directed percolation to be found by methods akin to those for the undirected problem.[3]

Cardy and Sugar[6] have shown that the pair connectedness for directed percolation may be obtained from the correlation function of a lattice model which, in the continuum limit, is in the same universality class as Reggeon field theory. This connection was important since it allowed the existing scaling theory, epsilon expansion, and numerical estimates of critical exponents for Reggeon field theory to be taken over for directed percolation. However, no exact values for critical exponents have been forthcoming by this route. The standard Potts model leads to a field theory, the fields of which have $\lambda-1$ components in contrast to the single component of Reggeon field theory. The restriction to odd flows and potential differences in our modified Potts models will, in the continuum limit, lead to new field theories which may be of interest in their own right as extensions of Reggeon field theory. Another difference between our work and that of Cardy and Sugar is that it applies to arbitrary directings and includes cyclic directings in addition to the standard directing in which all arcs have a positive component parallel to some preferred axis.

The simple extensions of the Potts model described here have much in common with the standard chiral Potts model and their study may provide information about the latter using universality. Even in the absence of exact solutions it will be possible to obtain much longer series expansions than for the standard chiral model since the expansion coefficients are flow polynomials. Nicolaides[7] has pointed out to us that the Kasteleyn-Fortuin relation for undirected percolation has been used to speed up Monte Carlo simulations of the standard Potts model.[8] He is planning to use our results to carry out similar simulations for chiral Potts models.

The interacting spin model we consider is a special case of the so-called $Z(\lambda)$ model[9] which is defined for our purposes on the directed graph H, having vertex set V and arc set A, as follows. With each vertex i of V we associate a state variable n_i, called its potential, which takes on the λ values $0,1,\dots,\lambda-1$. The Hamiltonian may be written in terms of the dimensionless interaction function h_a for the arc $a=(i,j)$ in the form, see Ref. 10,

$$\mathcal{H}=k_BT\sum_{a\in A}h_a(n_j-n_i)\,, \tag{1}$$

where the potential difference n_j-n_i is calculated mod $-\lambda$ and therefore $\mathcal{H}$ has Z_λ symmetry. The probability that vertices 1 and 2 are in states n_1 and n_2 depends only on the difference n_2-n_1 mod$(-\lambda)$. For $\beta=0,\dots,\lambda-1$, we therefore define

$$U_\beta(1,2;\lambda,H)=\mathrm{Prob}(n_2-n_1=\beta\bmod(-\lambda))=\frac{1}{Z(\lambda,H)}\sum_{n_1=0}^{\lambda-1}\cdots\sum_{n_v=0}^{\lambda-1}\delta_\lambda(n_2-n_1-\beta)\prod_{a\in A}e^{-h_a(n_j-n_i)}\,, \tag{2}$$

where $v=|V|$, $\delta_\lambda(\alpha)=1$ when $\alpha=0 \bmod(-\lambda)$ but zero otherwise, and the partition function $Z(\lambda,H)$ is defined so that

$$\sum_{\beta=0}^{\lambda-1} U_\beta(1,2;\lambda,H)=1\,. \tag{3}$$

If all the interactions are zero then $U_\beta(1,2;\lambda,H)$ has a value $1/\lambda$ for any β and the effect of the interaction is measured by the correlation function

$$\Gamma_\beta(1,2;\lambda,H)=U_\beta(1,2;\lambda,H)-1/\lambda\,. \tag{4}$$

Finally, we define the equivalent transmissivity vector $T_\alpha(1,2;\lambda,H)$ by the discrete Fourier transform

$$T_\alpha(1,2;\lambda,H)=\sum_{\beta=0}^{\lambda-1} e^{2\pi i\alpha\beta/\lambda}U_\beta(1,2;\lambda,H) \tag{5}$$

and from (3), $T_0(1,2;\lambda,H)=1$.

The pair interaction function $h_a(n_j-n_i)$ of the standard Potts model has only two values depending on whether vertices i and j are in the same state or different states; thus

$$h_a(\alpha)=\begin{cases}0, & \alpha=0,\\ \lambda K_a, & \alpha\neq 0\end{cases} \tag{6}$$

(the factor λ is included for consistency of notation with the usual Ising model which is given by $\lambda=2$) and the functions Γ_β and T_α defined above are independent of the directing of H and also have only two values depending on whether the subscript is zero or not zero,

$$\Gamma_\beta=\Gamma_1 \quad \text{for } \beta\neq 0 \tag{7}$$

and

$$T_\alpha=T_1 \quad \text{for } \alpha\neq 0\,. \tag{8}$$

For this model, Kasteleyn and Fortuin[1] have shown that

$$Z(\lambda,H)=\langle\lambda^\omega\rangle_{p_a=1-e^{-\lambda K_a}} \tag{9}$$

and

$$\Gamma_0(1,2;\lambda,H)=\frac{\lambda-1}{\lambda}\left(\frac{\langle\gamma_{12}\lambda^\omega\rangle}{\langle\lambda^\omega\rangle}\right)_{p_a=1-e^{-\lambda K_a}}, \tag{10}$$

or, equivalently,

$$U_1(1,2;\lambda,H)=\frac{1}{\lambda}\left(\frac{\langle(1-\gamma_{12})\lambda^\omega\rangle}{\langle\lambda^\omega\rangle}\right)_{p_a=1-e^{-\lambda K_a}}. \tag{11}$$

Here $\langle x\rangle_{p_a}$ denotes an undirected bond percolation average of the random variable x, where p_a is the probability that the arc a is open, and is given by

$$\langle x\rangle_{p_a}=\sum_{A'\subseteq A} x(H')\prod_{a\in A'}p_a\prod_{a\in A\setminus A'}(1-p_a)\,. \tag{12}$$

In this percolation model the direction of the arcs is ignored and an open arc (i,j) allows fluid to percolate from i to j, or from j to i. As is usual in undirected percolation, the subgraph H', with vertex set V and arc set A', partitions the vertices into clusters (connected components). The random variable ω is the number of clusters in a given configuration of open edges and γ_{12} has value 1 or 0 depending on whether or not the vertices 1 and 2 belong to the same cluster (i.e., there is a chain of open edges linking the vertices 1 and 2 in A').

More recently, Essam and Tsallis[11] showed that for the standard Potts model the Kasteleyn-Fortuin formulas could be rewritten in terms of an edge transmissivity variable t_a. For the general $Z(\lambda)$ model the arc transmissivity $t_a(\alpha)$ for the arc a is defined by (5) with H_a being the graph consisting of the single arc a. In this case

$$U_\beta(1,2;\lambda,H_a)=e^{-h_a(\beta)}/Z_a\,, \tag{13}$$

where

$$Z_a=\sum_{\alpha=0}^{\lambda-1} e^{-h_a(\alpha)}\,. \tag{14}$$

With this definition $t_a(0)=1$ and[12]

$$T_\alpha(1,2;\lambda,H)=\frac{1}{D(\lambda,H)}\sum_{\phi\in\mathcal{F}_{\alpha,\lambda}(H)}\prod_{a\in A}t_a(\phi(a))\,, \tag{15}$$

where $\mathcal{F}_{\alpha,\lambda}(H)$ is the set of integer $\bmod(-\lambda)$ rooted flows on H with an external flow of α in at vertex 1 and out at vertex 2 (i.e., a flow with value in the range $0,1,\ldots,\lambda-1$ is assigned to each arc of H such that the flow into any vertex is equal, $\bmod(-\lambda)$, to the outward flow). The denominator $D(\lambda,H)$ is such that $T_0(1,2;\lambda,H)=1$ and is given by a similar sum to that of (15) over the flows $\mathcal{F}_\lambda(H)=\mathcal{F}_{0,\lambda}(H)$ having no external flow. This result is an extension of earlier work of Biggs,[13] who showed that

$$Z(\lambda,H)=\lambda^{v-\epsilon}\left(\prod_{a\in A}Z_a\right)D(\lambda,H)\,, \tag{16}$$

where ϵ is the number of edges in H.

For the standard Potts model it follows from (6) that for $\alpha>0$,

$$t_a(\alpha)=t_a\equiv(1-e^{-\lambda K_a})/[1+(\lambda-1)e^{-\lambda K_a}]\,. \tag{17}$$

The result corresponding to (9) is, see Ref. 11,

$$D(\lambda,H)=\langle\lambda^c\rangle_{p_a=t_a}\,. \tag{18}$$

The random variable c is the cycle rank of the subgraph H' of H formed by the open bonds. Similarly, it was shown in Ref. 11 that instead of (10),

$$T_1(1,2;\lambda,H)=(\langle\gamma_{12}\lambda^c\rangle/\langle\lambda^c\rangle)_{p_a=t_a}\,. \tag{19}$$

Setting $\lambda=1$ we obtain from (11), (17), and (19)

$$T_1(1,2;\lambda=1,H)=1-U_1(1,2;\lambda=1,H) \tag{20}$$

$$=\langle\gamma_{12}\rangle_{p_a=1-e^{-K_a}} \tag{21}$$

which is the pair connectedness for undirected percolation with probability $1-e^{-K_a}$ that the arc a is open. These results are now extended to directed percolation.

We consider first a different $Z(\lambda)$ model whose transmissivity is expressible as a directed percolation average. The model is defined via its arc transmissivity by

$$t_a(\alpha)=\begin{cases}1, & \alpha=0,\\ t_a, & \alpha \text{ odd},\\ 0, & \alpha \text{ even},\end{cases} \tag{22}$$

where the flow α is directed parallel to a, and will be called the "odd-flow model." If λ is even, the functions U_β and T_α are again unchanged by reversing an arc since α and $\lambda-\alpha$ have the same parity.[12] This contrasts with the odd-λ case for which α and $\lambda-\alpha$ have different parity and U_β and T_α become dependent on the directing of H. For odd-λ, the model defined by (22) is therefore a chiral Potts model. The function $h_a(\alpha)$ for this model, obtained from $t_a(\alpha)$ by inverting the Fourier transform (5), has complex values and the resulting Hamiltonian is non-Hermitian. We note that $\lambda^{c(H')}$ in (18) is the number of integer mod$(-\lambda)$ flows on H' with no external flow and for the odd-flow model it may be shown that Eq. (18) is replaced by

$$D(\lambda,H)=\langle|\mathcal{F}_\lambda^{\mathrm{odd}}|\rangle_{p_a=t_a}, \tag{23}$$

where $\mathcal{F}_\lambda^{\mathrm{odd}}(H')$ is the set of these flows which are zero or odd for each $a\in A'$. It may also be shown that the equivalent vector transmissivity of this model is given by

$$T_\alpha(1,2;\lambda,H)=(\langle|\mathcal{F}_{\alpha,\lambda}^{\mathrm{odd}}|\rangle/\langle|\mathcal{F}_\lambda^{\mathrm{odd}}|\rangle)_{p_a=t_a}, \tag{24}$$

where now $\mathcal{F}_{\alpha,\lambda}^{\mathrm{odd}}(H')$ is defined in the same way as $\mathcal{F}_\lambda^{\mathrm{odd}}(H')$ but with an external flow α. The values of $|\mathcal{F}_\lambda^{\mathrm{odd}}(H')|$ for odd $\lambda\geq 3$ may be interpolated by a polynomial $F^{\mathrm{odd}}(\lambda,H')$ of degree at most $c(H)$.[14] The same is also true for even-λ values but a different polynomial is required, and we subsequently restrict attention to the chiral model with odd λ. $|\mathcal{F}_{\alpha,\lambda}^{\mathrm{odd}}(H')|$ is similarly interpolated by a polynomial $F_\alpha^{\mathrm{odd}}(\lambda,H')$ and it is shown in Ref. 14 that

$$F^{\mathrm{odd}}(1,H')=1 \quad\text{and}\quad F_1^{\mathrm{odd}}(1,H')=\pi_{12}(H'), \tag{25}$$

where

$$\pi_{12}(H')=\begin{cases}1, & \text{if there is a directed path from 1 to 2 in } H',\\ 0, & \text{if not}.\end{cases} \tag{26}$$

Thus, setting $\alpha,\lambda=1$,

$$T_1(1,2;1,H)=\langle\pi_{12}\rangle_{p_a=t_a}=C_{12}(t_a,H), \tag{27}$$

the pair connectedness for directed percolation, with probability p_a that the arc a is open, thereby extending the result (21) for the undirected model.

We emphasize at this point that the expression $|\mathcal{F}_{\alpha,\lambda}^{\mathrm{odd}}|$ used to obtain the transmissivity in (24) has no obvious meaning when $\lambda=1$. However, if in (24) we replace $|\mathcal{F}_{\alpha,\lambda}^{\mathrm{odd}}|$ by the interpolating polynomial $F_\alpha(\lambda,H)$, obtained for odd $\lambda\geq 3$, and then set $\lambda=1$, we obtain (27) by making use of the polynomial properties (25).

The extension of (20) is not so straightforward and requires consideration of the dual directed percolation problem of Dhar, Barna, and Phani.[15] In their DRP model, the arc $a=(i,j)$ is open in both directions with probability p_a but with probability $1-p_a$ the fluid can only percolate from i to j (i.e., is open parallel to the direction of the arc a). As usual the pair connectedness is the probability of finding an open path from 1 to 2 and will be denoted by $C_{12}^*(p_a,H)$. If there is a directed path from 1 to 2 in H then C_{12}^* has the value unity for all p_a just as the normal pair connectedness is identically zero when there is no such path. Percolation averages for this model will be denoted by $\langle\cdots\rangle_{p_a}^*$ and are defined as in (15) but now the graph H' corresponding to the arc set A' is obtained from H by replacing the arcs of A' by undirected edges. Thus H' is a partially directed graph.

We note that for the general $Z(\lambda)$ model, in analogy with (15),

$$U_\beta(1,2;\lambda,H)=\frac{1}{Z(\lambda,H)}\sum_{n\in\mathcal{P}_{\beta,\lambda}(H)}\prod_{a\in A}w_a(\delta n(a)), \tag{28}$$

where $\mathcal{P}_{\beta,\lambda}(H)$ is the set of state potentials such that $n_2-n_1=\beta$ mod$(-\lambda)$, δn is the potential difference obtained from the potential n, and

$$w_a(\alpha)=e^{-h_a(\alpha)}. \tag{29}$$

The partition function $Z(\lambda,H)$ is given by a similar summation with $\mathcal{P}_{\beta,\lambda}(H)$ replaced by

$$\mathcal{P}_\lambda(H)=\bigcup_{\beta=0}^{\lambda-1}\mathcal{P}_{\beta,\lambda}(H), \tag{30}$$

the set of potentials with no external constraint.

In order to extend (9) and (10) to directed percolation, we now consider a further case of the $Z(\lambda)$ model in which

$$w_a(\alpha)=\begin{cases}1, & \alpha=0,\\ e^{-\lambda K_a}, & \alpha \text{ odd},\\ 0, & \text{otherwise}.\end{cases} \tag{31}$$

We call this the "odd potential-difference model" or "odd PD model." As for the odd-flow model, when λ is even the functions U_β and T_α are again unchanged by reversing the directing of any subset of the arcs of H, whereas when λ is odd they become dependent on the directing of H which gives rise to a second chiral Potts model this time with a real Hamiltonian. It may be shown that for the odd potential-difference model Eq.

(9) is replaced by

$$Z(\lambda,H)=\langle|\mathcal{P}_\lambda^{\rm odd}|\rangle^*_{p_a=1-e^{-\lambda K_a}}, \qquad (32)$$

where, for a configuration in which the arcs $A'\subseteq A$ are open, $\mathcal{P}_\lambda^{\rm odd}(H')$ is the subset of $\mathcal{P}_\lambda(H)$ such that the potential difference is zero for $a\in A'$ (i.e., on the undirected edges of H') and is zero or odd for $a\notin A'$. For $\beta>0$,

$$U_\beta(1,2;\lambda,H)=(\langle|\mathcal{P}_{\beta,\lambda}^{\rm odd}|\rangle^*/\langle|\mathcal{P}_\lambda^{\rm odd}|\rangle^*)_{p_a=1-e^{-\lambda K_a}}, \qquad (33)$$

where the definition of $\mathcal{P}_{\beta,\lambda}^{\rm odd}(H')$ is similar to that of $\mathcal{P}_\lambda^{\rm odd}(H')$ with the additional condition that $n_2-n_1=\beta$ mod$(-\lambda)$. The values of $|\mathcal{P}_\lambda^{\rm odd}(H')|$ for odd $\lambda\geq 3$ may be interpolated by a polynomial $P^{\rm odd}(\lambda,H')$ of degree at most v, see Ref. 14, which takes the place of $\lambda^{\omega(H')}$ in (9). $|\mathcal{P}_{\beta,\lambda}^{\rm odd}(H')|$ may also be so interpolated and we denote the interpolating polynomial by $P_\beta^{\rm odd}(\lambda,H')$. It is shown in Ref. 14 that

$$P^{\rm odd}(1,H')=1 \quad\text{and}\quad P_1^{\rm odd}(1,H')=1-\pi_{12}(H'), \qquad (34)$$

where H' is the partially directed graph defined above, and it follows that

$$U_1(1,2;\lambda=1,H)=1-\langle\pi_{12}\rangle^*_{p_a=1-e^{-\lambda K_a}}$$
$$=1-C_{12}^*(1-e^{-\lambda K_a},H), \qquad (35)$$

which is the pair disconnectedness function for the dual directed percolation model.

Flows and potential differences are dual structures. For every flow on the directed plane graph H, there is a corresponding potential difference on the dual directed graph H^*.[16] Each arc a of H induces a dual arc a^* on H^* such that a^* is obtained from a by a clockwise rotation through a right angle. The corresponding flow ϕ and the potential difference δn satisfy $\phi(a)=\delta n(a^*)$. This gives a one-to-one correspondence between flows and potential differences on H and H^*.

Biggs[13] has used this correspondence to obtain a duality relation for the partition function for the general $Z(\lambda)$ model. The two chiral Potts models considered here form a dual pair in the sense of Biggs which implies that

$$Z^*(\lambda,H^*)=\langle|\mathcal{P}_\lambda^{\rm odd}|\rangle^*_{p_a=1-e^{-\lambda K_a^*}}$$
$$=\lambda\langle|\mathcal{F}_\lambda^{\rm odd}|\rangle_{p_a=e^{-\lambda K_a^*}}, \qquad (36)$$

where the starred average is calculated for the dual graph H^*. We can therefore realize the dual flow model as a potential model with an appropriate connectivity. This duality result for the partition function may be also extended to the transmissivity and correlation functions since (36) is also valid for $\mathcal{P}_{\alpha,\lambda}^{\rm odd}$ and $\mathcal{F}_{\alpha,\lambda}^{\rm odd}$ provided that the roots 1^* and 2^* are chosen as follows. Let the roots of H be joined by an additional arc (2,1). Let 1^* and 2^* be the roots of the dual graph H^* corresponding, respectively, to the finite and infinite faces formed by the addition of the arc (2,1). Then the arcs (2,1) and $(2^*,1^*)$ are dual arcs and form a clockwise pair.

Setting $\lambda=1$ in (36) and its rooted analog and using (24), (31), and (35), we obtain

$$C_{1^*2^*}^*(p_a,H^*)=1-C_{12}(1-p_a,H), \qquad (37)$$

which extends the Dhar, Barna, and Phani[15] result for the percolation probability to the pair connectedness and to arbitrary dual pairs of planar graphs. Also this equation may be obtained without considering the $Z(\lambda)$ model by using the correspondence between configurations on H and H^* in which a closed edge of H corresponds to a two-way open edge of H^*. If H'^* is the partially directed graph corresponding to the subgraph H' then there is directed path from 1 to 2 on H' if and only if there is no such path from 1^* to 2^* on H'^* and hence $\pi_{1^*2^*}(H'^*)=1-\pi_{12}(H')$. Equation (37) follows by noting that if p_a is the probability of a two-way open edge on H^*, then using the correspondence between configurations the probability of an open edge on H is $1-p_a$.

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