Chaos in the Low-Lying Collective States of Even-Even Nuclei

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(Received 22 May 1990; revised manuscript received 27 September 1990)

We study the fluctuation properties of the spectrum and the electromagnetic $E2$ transitions of low-lying collective states in even-even nuclei. Using the framework of the interacting-boson model we discuss the transition between rotational and γ-unstable nuclei. Near those two limits the system exhibits regular behavior but in the transition region it shows chaotic behavior, where the fluctuations are characterized by the Gaussian orthogonal ensemble. Analysis of the classical mean-field dynamics of the five nuclear quadrupole shape degrees of freedom confirms the quantal results.

PACS numbers: 21.10.Re, 05.45.+b, 21.60.Fw, 23.20.−g

Random-matrix theories[1] (RMT) have been useful in the study of the fluctuation properties of neutron resonances in heavy nuclei.[2] Their use was justified by the complexity of the nuclear system and the large numbers of degrees of freedom. In recent years, however, it was conjectured that the validity of random-matrix theory can be extended to quantal systems with few degrees of freedom when the underlying classical motion is chaotic.[3] In particular, the Gaussian orthogonal ensemble (GOE) of random matrices has been associated with chaotic systems which are time-reversal symmetric. The conjecture was confirmed in numerous studies, mostly of model systems in two degrees of freedom.[3–5]

On the experimental side, the most complete data available are the nuclear data ensemble[6] (NDE) consisting of neutron and proton resonances. These resonances are in regions of high nuclear level density and are consistent with the GOE predictions. The question which arises is whether such a chaotic behavior could prevail also in the low-lying collective part of the nuclear spectrum. Experimentally known low-lying levels in various nuclei were analyzed in Ref. 7. Because of the small sample, it was necessary to group together levels with different spin-parity and only partial conclusions could be reached.

The purpose of this Letter is to study the transition from regular to chaotic motion in low-lying collective states of nuclei by using a realistic theoretical model. With few exceptions, such as that of Rydberg atoms in strong diamagnetic field,[8] most of the model problems studied in quantum chaos were unrealistic. It is important to investigate whether the signature of quantum chaos observed in the two-dimensional systems also prevails in realistic nuclear models where the number of relevant degrees of freedom is larger than two but still much less than that of the compound nucleus (~ 100) where the neutron resonances were observed.

Most studies of quantum chaos were restricted to the fluctuation properties of the spectrum alone. A study of transition intensities provides an additional probe.[9,10] In the present Letter we study simultaneously the fluctuations of the spectrum and of the electromagnetic $E2$ transitions.

A complete understanding of the character of the dynamics requires the study of the classical limit. This is considerably more difficult in our case than the usual two-dimensional studies since the number of degrees of freedom is five, corresponding to the five nuclear quadrupole shape parameters. Nevertheless, we are able to accomplish such a study using Monte Carlo techniques. The classical limit here is also different from most other studies in that it actually describes the mean-field evolution of the many-body quantal system.[5]

The model that we use is the interacting-boson model[11] (IBM). It is suitable, in particular, for our study since it provides realistic spectra and electromagnetic transition intensities, yet as an algebraic model it is relatively simple to solve. Algebraic models have another property which is particularly useful here. The completely integrable Hamiltonian (for which the motion is necesarily regular) is relatively easy to identify as the one which possesses a dynamical symmetry. Such a symmetry occurs when the algebraic Hamiltonian H can be expressed as a function of Casimir invariants C of a chain of subalgebras of the original algebra G:

$$H=\alpha C(G)+\alpha' C(G')+\alpha'' C(G'')+\cdots, \qquad G\supset G'\supset G''\supset\cdots. \tag{1}$$

These Casimir invariants form a set of constants of the motion in involution. If this set is not complete, there are missing labels which can be related to invariants of subalgebras which are not Casimir invariants. With these invariants we obtain a complete set of constants and the system is completely integrable.

The interacting-boson model where $G\equiv \mathrm{U}(6)$ has three such dynamical symmetries[11]

$$\mathrm{U}(6)\supset\left\{\begin{matrix}\mathrm{U}(5)\supset\mathrm{O}(5)\\ \mathrm{SU}(3)\\ \mathrm{O}(6)\supset\mathrm{O}(5)\end{matrix}\right\}\supset\mathrm{O}(3)\;\begin{matrix}(\mathrm{I})\\ (\mathrm{II})\\ (\mathrm{III})\end{matrix}\,. \tag{2}$$

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Chain (I) describes vibrational nuclei, chain (II) rotational nuclei, and chain (III) γ-unstable nuclei. In this paper we shall study a family of Hamiltonians which correspond to a transition between chains (II) and (III), describing deformed nuclei which are gradually becoming softer in the γ direction:

$$H = E_0 + c_1 \mathbf{L}^2 + c_2 Q^\chi \cdot Q^\chi . \tag{3}$$

In (3) $\mathbf{L}$ is the angular momentum and Q^χ is a quadrupole operator,

$$Q^\chi = (d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d})^{(2)} + \chi (d^\dagger \times \tilde{d})^{(2)} , \tag{4}$$

which depends on a parameter χ. E_0, c_1, and c_2 are constants. When $\chi = -\sqrt{7}/2$ we get the SU(3) Hamiltonian and when $\chi = 0$ we obtain the O(6) one. The $E2$ transition operator[11] is taken to be proportional to Q^χ,

$$T(E2) = \alpha_2 Q^\chi . \tag{5}$$

We have studied the chaos of the spectra and the $E2$ intensities as a function of χ, for several fixed values of the spin-parity. A study of spectral fluctuations alone using the IBM in the SU(3) and O(6) limits was presented in Ref. 12 but did not include the transition region.

We have used two statistical measures to determine the fluctuation properties of the unfolded energy levels:[3-5] the nearest-neighbor level-spacing distribution $P(S)$ and the Δ_3 statistics of Dyson and Metha.[3] The level spacing is expected to be the Poisson distribution $P(S) = e^{-S}$ for a regular system and the Wigner (GOE) distribution $P(S) = (\pi/2) S \exp(-\pi S^2/4)$ for a chaotic one. The Δ_3 statistics measures the spectral rigidity. For the Poisson statistics $\bar{\Delta}_3(L) = L/15$ and for the much stiffer GOE spectrum we have for large L, $\bar{\Delta}_3(L) \approx \pi^{-2} \ln L - 0.0007$. We have analyzed levels within each spin-parity class of the model (3) for $J^\pi = 0^+, 2^+, 3^+, \dots, 8^+$ using $N = 20$ bosons. Figure 1 shows for several values of χ the level-spacing histograms and the Δ_3 statistics for the 121 levels with $J = 8^+$. We have fitted the level-spacing distribution with a Brody distribution[13] (solid lines) of the form

$$P_\omega(S) = A S^\omega \exp(-\alpha S^{1+\omega}) , \tag{6}$$

where α and A are chosen such that P is normalized and $\langle S \rangle = 1$. The distribution (6) describes intermediate situations between the Poisson ($\omega = 0$) and Wigner ($\omega = 1$) distributions. We see that near the two dynamical-symmetry limits $\chi = -\sqrt{7}/2$ and $\chi = 0$, the behavior of $P(S)$ and $\Delta_3(L)$ is close to Poisson, as it should be for integrable systems. However, as χ moves away from those two limits the statistics changes gradually to the GOE one, which indicates the onset of chaotic motion.

The intensity distributions[9,10] can offer a more sensitive probe. We have analyzed $E2$ transitions following the procedure of Ref. 10. The $B(E2)$ value between an initial state i and a final state f, which we denote by y, is given in terms of the reduced matrix elements of $T(E2)$:

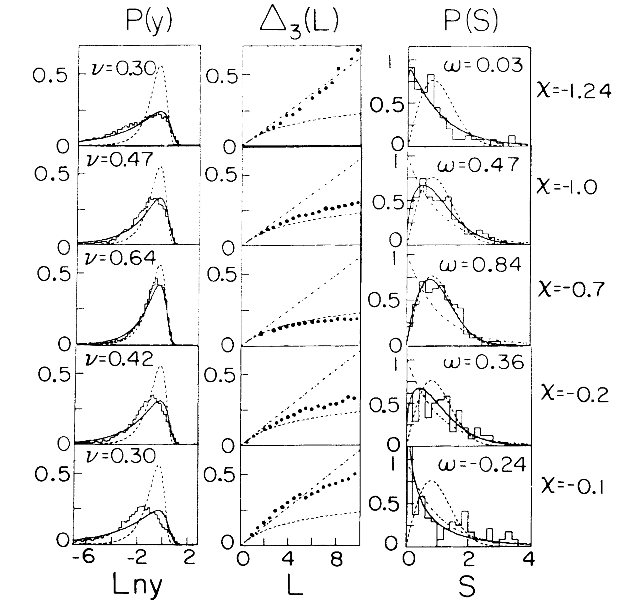

FIG. 1. The nearest-neighbor level-spacing distribution $P(S)$ (histograms in the right-hand column), the Dyson-Metha Δ_3 statistics (dots in the middle column), and the $B(E2)$ distribution $P(y)$ (histograms in the left-hand column) of the $J = 8^+$ states for several values of the parameter χ, describing the transition from rotational nuclei ($\chi = -\sqrt{7}/2$) to γ-unstable nuclei ($\chi = 0$). The corresponding Hamiltonian is (3) with $c_1 = 0$ and $c_2 = -0.1$, the number of bosons is $N = 20$, and the $E2$ operator is (5). The dashed lines describe the GOE limit (Wigner) and the dash-dotted lines are the Poisson distribution. The solid lines (in the right-hand column) are the best fits with the Brody distribution (6) with the ω quoted. In the left-hand column [of $P(y)$] the dashed lines are the Porter-Thomas distributions (8) which have the same $\langle y \rangle$. Notice that a logarithmic scale is used for the $E2$ intensity y. The solid lines are the best fit with the χ^2 distribution (9) in ν degrees of freedom.

$$y \equiv B(E2; i \to f) = \frac{1}{2J+1} |\langle f || T(E2) || i \rangle|^2 . \tag{7}$$

We have analyzed $E2$ transitions $J^\pi \to J^\pi$ within a given spin-parity class. To separate the nonuniversal smooth behavior of $B(E2)$ with energy we divide each y by an average intensity calculated[10] using Gaussians of width γ centered around each unfolded level. The renormalized intensities (denoted also by y in the following) are then used to construct histograms of their distribution $P(y)$ such that $P(y)dy$ measures the probability to find an intensity in the interval dy around y.

The GOE prediction for $P(y)$ is the Porter-Thomas distribution[2]

$$P(y) = (2\pi \langle y \rangle)^{-1/2} \exp(-y/2\langle y \rangle) . \tag{8}$$

To describe deviations from the Porter-Thomas distribution, we use the χ^2 distribution in ν degrees of freedom

($\nu > 0$) introduced in Ref. 9,

$$P_\nu(y) = A y^{\nu/2-1} \exp(-\nu y / 2\langle y\rangle) , \tag{9}$$

with $A = (\nu/2\langle y\rangle)^{\nu/2}/\Gamma(\nu/2)$. For $\nu = 1$ we obtain the Porter-Thomas distribution.

The left-hand column of Fig. 1 shows the distributions $P(y)$ (histograms) for the $8^+ \rightarrow 8^+$ $E2$ transitions (there are 7260 such transitions). The dashed lines are the Porter-Thomas distributions which have the same $\langle y\rangle$ as the actual distributions. The best fits with (9) are the solid lines. Notice that we have used a logarithmic scale for the intensities y since the weak transitions range over several orders of magnitude. In Ref. 10 it was found that when the system becomes more regular the value of ν decreases from 1 towards 0. In Fig. 1 we see that near the SU(3) and O(6) limits, where the nuclear dynamics is regular, ν obtains small values (≈ 0.3). Indeed in these limits there are selection rules which make a few allowed transitions large and many others forbidden. For example, in the SU(3) limit only $E2$ transitions between levels which belong to the same SU(3) representation are allowed. Near the limits these forbidden transitions are very weak and as a result the distribution is relatively broad with a wide range of weak transitions. For intermediate values of χ, the $E2$ distributions get closer to the Porter-Thomas one and the maximal $\nu \approx 0.7$ is obtained for $\chi \approx -0.6$. Thus there is a strong correlation between the spectral fluctuations and the intensity fluctuations. However, even for $\chi = -0.6$ the $B(E2)$ distribution shows deviations from the Porter-Thomas one. This means that the motion is not yet completely chaotic. Indeed the character of the dynamics can be energy dependent. A study of the classical limit is capable of providing information on this energy dependence.

The semiclassical limit[14] of the IBM is a mean-field approximation where $1/N$ plays the role of $\hbar$ and is obtained by using boson condensates

$$|\boldsymbol{\alpha}\rangle = \exp(-|\boldsymbol{\alpha}|^2/2) \exp\left(\alpha_s s^\dagger + \sum_\mu \alpha_\mu d_\mu^\dagger\right)|0\rangle . \tag{10}$$

The six complex numbers $\boldsymbol{\alpha}$ together with $i\boldsymbol{\alpha}^*$ are then canonical-conjugate variables for the classical Hamiltonian $\mathcal{H} \equiv \langle\alpha|H|\alpha\rangle$. The boson-number conservation $\alpha_s^*\alpha_s + \sum \alpha_\mu^*\alpha_\mu = N$ is a constraint which effectively reduces the number of degrees of freedom to five. These are the five nuclear quadrupole shape parameters. The classical limit is obtained for $N \rightarrow \infty$.

We have used Monte Carlo techniques to determine the fraction of the chaotic volume σ of a given energy-angular-momentum surface in phase space. A chaotic trajectory is characterized by a positive maximal Lyapunov exponent λ, while a regular trajectory has $\lambda = 0$. Figure 2 shows a classical phase diagram in the χ-ϵ (energy per boson) plane for spin $J = 8$. $\epsilon_{\min}$ represents the lowest possible energy for a given value of χ. The other two solid curves are contours of $\sigma = 0.1$ and 0.9, separating the regular regime ($\sigma < 0.1$) from the completely chaotic one ($\sigma > 0.9$). Near $\chi = -\sqrt{7}/2$ and $\chi = 0$ the classical motion is regular at all energies. In the intermediate regime ($-0.8 \lesssim \chi \lesssim -0.4$) the motion is regular only at very low energies (near $\epsilon_{\min}$) and chaos sets in rapidly over a narrow transition region. These results are in good accord with the quantal analysis presented above.

To summarize our results we show in Fig. 3 both quantal and classical measures of chaos versus χ for $J^\pi = 2^+, 4^+, 6^+$, and 8^+. In the left-hand column we show the values of ω (characterizing the level-spacing

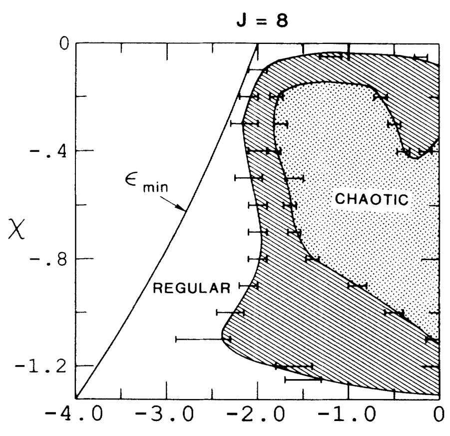

FIG. 2. Phase diagram in the χ-ϵ plane for $J = 8$. $\epsilon_{\min}$ is the minimal energy at a given χ. The other two solid lines represent contours where the chaotic volume is 10% and 90% of the total-energy-angular-momentum surface. The bars are the statistical Monte Carlo errors and the lines through them are just to guide the eye.

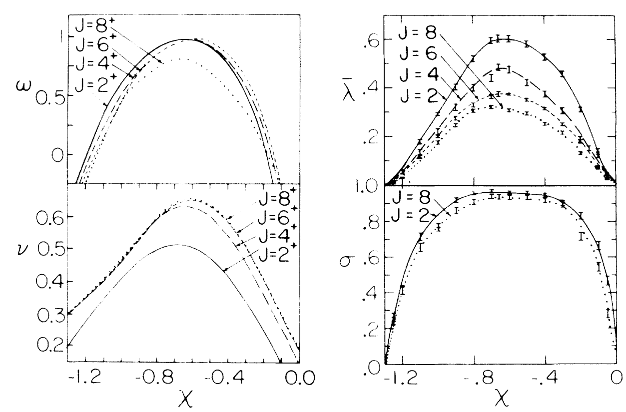

FIG. 3. Left-hand column: quantal measures of chaos; the level-spacing distribution parameter ω vs χ (top) and the number of degrees of freedom ν of the $J^\pi \rightarrow J^\pi$ $B(E2)$ intensity distribution vs χ (bottom) for $J = 2^+, \ldots, 8^+$. Right-hand column: classical measures of chaos; the average maximal Lyapunov exponent $\bar{\lambda}$ vs χ (top) for $J = 2$, 4, 6, and 8 and the chaotic fractional volume σ vs χ (bottom) for $J = 2$ and 8.

distribution) and v [characterizing the $B(E2)$ distribution]. In the right-hand column we have plotted two classical measures: the average maximal Lyapunov exponent $\bar{\lambda}$ and the chaotic fractional volume σ (calculated irrespective of the energy). All measures are well correlated; they indicate regular motion near $\chi = -\sqrt{7}/2$ and $\chi=0$ and maximal chaos around $\chi \sim -0.6$. It is seen from Fig. 3 that the onset of chaos is somewhat more gradual when approached from the SU(3) side. This observation is also confirmed in Fig. 2 where the transition region near the SU(3) limit is considerably wider than the one near the O(6) limit. In the SU(3) limit of rotational nuclei there is a good quantum number K, the projection of the spin on the nuclear symmetry axis. As χ increases from its SU(3) value, K becomes only an approximate quantum number and eventually it is completely broken when the chaotic region in χ is reached. The fact that this transition is more gradual than the one on the O(6) side is an indication that it is harder to break K than the corresponding constant of the O(6) limit. In the exact SU(3) limit, where K appears as a missing label, the energy is independent of K and levels of a given spin which belong to the same SU(3) representation but with different K values are degenerate. This K degeneracy describes an overintegrable situation where the spacing distribution exceeds the Poisson distribution near $S \approx 0$ (notice that the Brody parameter ω becomes negative). The O(6) limit has a missing label too ($\tilde{\nu}_\Delta$) but the lowest spin for which it causes degeneracy is $J=6$. For the $J=8^+$ levels we see in Fig. 1 that for $\chi = -0.1$, Δ_3 lies above the "regular" line $L/15$.

Another effect seen in Fig. 1 is the saturation of the Δ_3 statistics, namely, the flattening out at a finite nonuniversal L. This effect, not predicted by RMT, was explained by Berry[15] using semiclassical arguments. The value of L where saturation occurs is determined by the period of the shortest classical orbit and must have a certain scaling with N which is confirmed in our studies.

To conclude, we have observed the onset of chaos in both the level and $B(E2)$ statistics in the transition region between rotational and γ-unstable even-even nuclei. We have also shown that the classical measures of chaos are strongly correlated with the quantal ones.

We acknowledge interesting discussions with F. Iachello, I. Talmi, and R. Gilmore. One of us (A.N.) acknowledges the hospitality of the Center for Theoretical Physics at Yale University. This work was supported in part by the Department of Energy Contract No. DE-AC02-76ER3074. Y.A. is an A.P. Sloan fellow.

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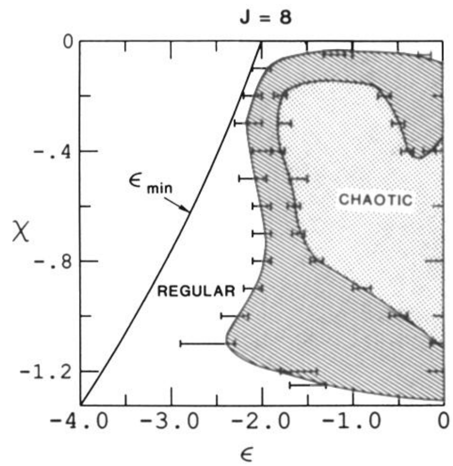

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