Light-Front Tamm-Dancoff Field Theory

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(Received 6 August 1990)

Light-front theory may provide a promising avenue of research for nuclear and particle physics, but a Tamm-Dancoff truncation of field theory is required for practical computations. Such a truncation limits the number of virtual mesons allowed in hadronic field theories, or the number of quarks and gluons allowed in bound states described by quantum chromodynamics. Past Tamm-Dancoff renormalization problems are analyzed and a solution is proposed.

PACS numbers: 11.10.St, 11.10.Ef, 11.10.Gh, 11.10.Qr

Despite years of effort, strongly interacting relativistic systems are not understood. We are able to compute the properties of strongly interacting nonrelativistic systems using traditional methods from many-body quantum mechanics. Bound and scattering states of weakly interacting particles are well described by perturbative field theory. The major unsolved problem is that of the highly relativistic bound state. The main difficulties are far better understood now than they were in the 1940s when the effort to use field theory in the study of strong interactions began, but no practical tool has been developed for circumventing these difficulties. In this Letter we propose a path that leads around some of these problems and hopefully through the remainder, light-front Tamm-Dancoff (LFTD). LFTD is simply the original Tamm-Dancoff approach[1,2] applied to light-front field theory.[3] The most closely related work is that of Brodsky, Lepage, Pauli, and collaborators.[4]

Two key areas where the relativistic bound-state problem is central are nuclear physics and quantum chromodynamics (QCD). Consider first the problem of understanding the structure of light nuclei. At low energy and low resolution we can eliminate intermediate- and high-energy degrees of freedom, and describe nuclei using nonrelativistic nucleons interacting via potentials. As energy and resolution are increased we believe a limit is approached in which nuclei are systems of many highly correlated quarks and gluons, but there are many ways that this limit might be reached. In particular, there might be an intermediate regime where nonrelativistic models prove inadequate, but where relatively few hadronic degrees of freedom can be utilized to accurately describe both nuclear structure and response.[5] To determine if this is the case, one must have sufficiently accurate descriptions of strongly interacting hadronic systems. One-pion exchange is usually considered to be adequately described by potentials, so one really wants to push the description at least to the range of two-pion exchange. In this range it is not reasonable to consider pion exchange without including the fact that pions dress nucleons; nor does it make any sense to ignore the fact that pions interact with one another strongly. All strongly interacting degrees of freedom included in any problem should be allowed to fully interact with one another. Hamiltonian methods are extremely effective in describing systems of a few strongly interacting particles, and are immediately suggested by this problem.

The second example is the problem of bound states of light quarks in QCD. The constituent quark model was invented for this problem, and after a complicated history led to the development of QCD; however, we are little closer to building the bridge between these two extremes than we were in the early 1970's. Lattice QCD is not adequate for this task yet because of the severe constraints placed by the need to use uniform grids.[6] At present a single lattice of equally spaced points is forced to span all distance scales appropriate to a bound state. What one prefers here is a method that leads to a sequence of descriptions intermediate between the constituent quark model and QCD. This suggests to us a series of effective Hamiltonians governing the behavior of quasiparticle quarks and gluons. At low resolution one sees constituent quarks interacting via confining potentials. As resolution increases, one begins to see additional constituent gluons. As resolution increases further, one should begin to resolve the structure of constituent quarks and gluons, and their interactions should approach the simple point coupling of QCD. Whether it is possible to build a sufficient number of layers of structure to actually reach the QCD Hamiltonian, with pointlike quarks and gluons, is an open research question. The point is that quasiparticles and effective interactions provide a promising way to build levels of structure without having to describe every level using the grid appropriate to the finest level of structure; and again one is led to consider Hamiltonian methods and a renormalization-group approach.[7]

The power of Hamiltonian methods is well known from the study of nonrelativistic many-body systems. Why have they not been elaborated in the study of relativistic field theory? Consider a typical Hamiltonian approach, in which one chooses a physically motivated finite basis and diagonalizes the Hamiltonian within the subspace spanned by that basis. This is precisely the approach first developed for field theory by Tamm,[1] and independently discovered by Dancoff,[2] both of whom sought a relativistic equation for the deuteron. Even in nonrelativistic systems this approach fails when one starts with a noninteracting ground state that is very far removed from the real ground state; and in equal-time field theory the noninteracting and interacting vacua are orthogonal. In practical terms, one tries to study particles built on top of the vacuum only to find that every amplitude is dominated by "disconnected" vacuum pieces. In perturbation theory there are analytic methods for solving this problem, but no such methods exist for the Hamiltonian calculation.

The vacuum problem was not the only source of difficulty for Tamm and Dancoff in equal-time field theory,[8] and many people investigated Tamm-Dancoff theory during the 1950s. They gave up when, amongst other problems, it became apparent that the Tamm-Dancoff truncation prohibits covariant renormalization beyond lowest order. It should not be surprising that such problems occur, because Lorentz-boost operators contain interactions that change particle number in equal-time theories.[9]

These problems are either averted or redefined in LFTD when one makes a crucial observation; *renormalization requires counterterms to depend on the sector of Fock space within which they act.* Sector-dependent renormalization violates locality but is needed to compensate for nonlocalities arising from the Tamm-Dancoff approximation itself.

Now consider how past difficulties appear in light-front quantization. On the light front the vacuum is trivial, and the original vacuum problem simply does not appear. This triviality results from the fact that light-front longitudinal momenta cannot be negative. One pays two prices for this triviality. The first price is that infrared (i.e., small longitudinal momenta) divergences are more severe on the light front than they were in equal time, leading to "spurious" divergences. The second price is that symmetry breaking[10] and the Goldstone phenomenon are usually said to proceed via "vacuum" condensation, and it is not obvious how the symmetry-breaking phase transition can occur when the vacuum is trivial. The solution to this apparent dilemma is simple, because even in light-front field theory the vacuum becomes degenerate at a critical point and the ground state becomes a superposition of states. Past the critical point one must discover the effective Hamiltonian in the broken-symmetry phase, and in this phase the new vacuum is again trivial. The use of the light front does not make the problem of finding the Hamiltonian any easier, but it also does not make it any more difficult. An essential question for QCD is what new effective interactions are induced by the chiral phase transition.

We will use a simple example to illustrate LFTD. Given a second-quantized Hamiltonian, begin by truncating Fock space and expanding the wave function for the state of interest in the remaining space. The equation to be solved is the second-quantized version of Einstein's equation,

$$P^2|\Psi\rangle = 2P^+P^-|\Psi\rangle = M^2|\Psi\rangle ,$$

where P^+ is the light-front "momentum" operator and P^- is the light-front "Hamiltonian." This equation is projected onto a complete set of states within the truncated Fock space, typically resulting in a set of coupled integral equations that determine the eigenvalues and eigenstates.

The basic features of LFTD are most simply discovered in super-renormalizable theories, where the ultraviolet renormalization problem is trivial. We will use the Yukawa model in $1+1$ dimensions, and drastically simplify the Hamiltonian by dropping antifermions and instantaneous interactions that do not enter the lowest-order calculations. The complete Hamiltonian can be found in Refs. 11 and 12. When antifermions are dropped, the only divergences are spurious.[11] There is no need for the approximations we make to be valid, as the example is merely intended to elucidate the method. Our simplified Hamiltonian is

$$P^- = \int \frac{dk}{2\pi 2k} a(k)^\dagger a(k) \frac{m_B^2}{2k} + \int \frac{dk}{2\pi k} b(k)^\dagger b(k) \frac{1}{2k}\left[m_F^2 + \frac{\lambda^2}{4\pi}\beta(k)\right]$$
$$+ \frac{\lambda m_F}{4\pi}\int \frac{dk_1}{\sqrt{k_1}}\int \frac{dk_2}{2\pi 2k_2}\int \frac{dk_3}{\sqrt{k_3}}\delta(k_1-k_2-k_3)\left[[b(k_1)^\dagger b(k_3)a(k_2) + b(k_3)^\dagger b(k_1)a(k_2)^\dagger]\left[\frac{1}{k_1}+\frac{1}{k_3}\right]\right],$$

where

$$\beta(k) = \mathcal{P}\int_{x_B P^+}^{\infty} \frac{dq}{q}\,\frac{k}{k-q} .$$

The boson creation operator is $a^\dagger$, and the fermion creation operator is $b^\dagger$. $\mathcal{P}$ represents the principal value and β, which arises from normal ordering, is called the self-induced inertia. We regulate the theory by removing all bosons whose momenta do not satisfy $x_B P^+ < p$, and all fermions whose momenta do not satisfy $x_F P^+ < p$. Note that the cutoff employed depends on the total momentum of the state, which is conserved. By choosing the cutoff in this fashion, one avoids explicitly breaking covariance. Other cutoffs and definitions of the self-inertia are possible.

First consider charge-one states with momentum P, and truncate Fock space so that only states containing a bare fermion, or a bare fermion and one boson, are retained. It is also convenient to scale the momentum P out of the problem by changing variables to momentum fractions, $k_i = x_i P$. With the above choice of cutoffs, this rescaling completely removes P from the problem. We can present only the final results, which are crudely indicative of the equations one finds in any LFTD calculation. Expanding the wave function in this space and projecting Einstein's equation leads to two coupled equations,

$$\left[M^2 - m_{F0}^2 - \frac{\lambda^2}{4\pi}\beta(1)\right]c_0 = \lambda m_F \int_{x_B}^{1-x_F} \frac{dx}{\sqrt{4\pi x}}\left[1 + \frac{1}{1-x}\right]c_1(1-x;x)$$

and

$$\left[M^2 - \frac{m_{F1}^2}{1-x} - \frac{m_B^2}{x}\right]c_1(1-x;x) = \lambda m_F \frac{1}{\sqrt{4\pi x}}\left[1 + \frac{1}{1-x}\right]c_0,$$

where the rescaled self-inertia is

$$\beta(x) = \mathcal{P}\int_{x_B}^{\infty} \frac{dy}{y}\frac{x}{x-y}.$$

c_0 is the amplitude for finding a bare fermion and $c_1(1-x;x)$ is the amplitude for finding a bare fermion with momentum fraction $1-x$ and a boson with momentum fraction x. The eigenvalue M^2 depends on the state. There should be a complete set of fermion-boson scattering states, for which $M^2 \geq (m_F + m_B)^2$, with m_F being the physical fermion mass. In addition, there should be a single state corresponding to the physical fermion, with eigenvalue $M^2 = m_F^2$. We have indicated that the bare mass depends on the sector of Fock space, using m_{F0}^2 in the lowest sector and m_{F1}^2 in the second sector. Note that β does not occur in the second equation. This is accomplished either by absorbing β into m_{F1}^2 or by simply dropping it. Above the scattering threshold, a Lippmann-Schwinger approach is required; however, it is easy to see that setting $m_{F1}^2 = m_F^2$ yields the correct scattering threshold.

Having completed mass renormalization in the second sector of Fock space, consider the state below the scattering threshold, the physical fermion. For this state use the second LFTD equation to eliminate the amplitude c_1 in the first equation, leading to the eigenvalue equation

$$m_{F0}^2 = m_F^2 - \frac{\lambda^2}{4\pi}\beta(1) + \frac{\lambda^2}{4\pi}m_F^2\int_{x_B}^{1-x_F}\frac{dx}{1-x}\frac{(2-x)^2}{x^2 m_F^2 + (1-x)m_B^2}.$$

Both $\beta(1)$ and the integral contain divergences, but these cancel.[11] We can now use this equation to fix the bare mass. The renormalization condition guarantees that $M^2 = m_F^2$ is a solution of the original equation.[13] This counterterm can now be used in higher orders of Tamm-Dancoff theory where additional bosons are allowed, and when many-fermion states are studied. In the former case this first "mass" counterterm moves from the lowest sector of Fock space to the sector with one boson, and one must determine a new mass counterterm for the lowest sector. In each order of Tamm-Dancoff theory old counterterms migrate to new sectors of Fock space, and the mass-renormalization condition determines a new counterterm in the lowest sector. This procedure leads to the correct scattering thresholds for asymptotic states containing only free fermions and bosons, even when the asymptotic particles become dressed.

To see how counterterms become nested, consider the two-fermion problem. Truncate Fock space so that only states with two bare fermions and one extra boson are allowed. The LFTD integral equations, which we do not have space to show,[12] couple the sector with two bare fermions to the sector with an additional boson. Following steps directly analogous to those above, one can eliminate the latter sector below boson production threshold to obtain a two-body equation for the fermions. The fermions are dressed by bosons and interact by one-boson exchange in this approximation. Spurious divergences arise; however, if one uses the mass counterterm computed for the single fermion, all divergences cancel and the two-fermion scattering threshold is properly fixed at $4m_F^2$. Our renormalization procedure manages to save this aspect of cluster decomposition by insisting that a fermion which cannot dress itself propagates with the physical mass, and the proper mass for other cases is dependent on the degree to which the fermion can dress itself. We do not discuss other aspects of cluster decomposition that are far more subtle.

The two-fermion LFTD equation has several nice features. It is unitary, covariant, and finite for all values of the cutoffs. In $1+1$ dimensions, this is true for arbitrary truncations of Fock space. It is possible to show that in the limit of large masses this equation reduces to a Schrödinger equation for two fermions interacting via a static Yukawa potential,[12] as one would expect. The utility of such equations for the study of few-nucleon bound states with interactions at least partially mediated by meson exchange should be obvious.

Another potential advantage of light-front field theory is that none of the boost operators contain interactions;[9] therefore, the truncation of Fock space by particle number (e.g., the removal of all states containing more than two pions) does not violate boost invariance, although in $3+1$ dimensions it does violate rotational invariance. As a result, when one studies *perturbation* theory,[11] it is possible to derive boost-invariant results within a truncated Fock space (e.g., with antifermions removed) after using perturbative renormalization and removing all cutoffs. A far more difficult question to answer is what happens when all cutoffs are removed in the nonperturbative Tamm-Dancoff approximation. This is a nonperturbative renormalization problem, and since the failure of Tamm-Dancoff theory in the 1950s it has been recognized that such problems need not be solved by the same counterterms that are adequate in perturbation theory.[7] In lowest order, LFTD closely resembles perturbation theory, and it is easy to see that mass renormalization is sufficient to remove all dependence on cutoffs as they approach their limits. It remains an open problem to determine what happens in higher orders.

There are two truncations in LFTD, the Tamm-Dancoff truncation that limits particle number and cutoffs that regulate the theory. Both of these truncations are computational necessities, and there are computational limits governing the extent to which these truncations can be removed. If the degrees of freedom that are discarded are actually not important, a straightforward application of LFTD should converge rapidly. However, in most problems of interest, one must derive effective interactions induced by the neglected degrees of freedom, using a renormalization-group approach to obtain convergence. The dominant effect of weakly coupled high-energy states is to alter the masses and couplings of the low-energy states.[7] When the high-energy states couple strongly, one must analyze renormalization on a case-by-case basis because no reliable generic methods exist. We are hoping that asymptotic freedom will simplify the analysis for QCD; however, in effective hadronic theories one is always faced with a strong-coupling problem. LFTD may allow one to explore this problem more thoroughly than other methods, but it does not solve the problem.

It is easy to construct a long list of difficult and interesting problems that we have not mentioned above, including an important list for gauge theories, some of which show up in perturbation theory.[14] Some of these problems must be solved before LFTD can be effectively applied to the nuclear- and particle-physics problems described above; however, LFTD has already overcome the most serious difficulties that crippled Tamm and Dancoff in equal-time theories. This method offers one of the only promising lines of attack on the difficult problem of relativistic bound states and should be thoroughly explored.

We gratefully acknowledge useful discussions with Stan Brodsky, Charlotte Elster, Stan Glazek, Kent Hornbostel, Peter Lepage, Daniel Mustaki, Steve Pinsky, Mikolaj Sawicki, Junko Shigemitsu, and James Vary. This work was supported in part by the National Science Foundation under Grants No. PHY-8719526 (R.J.P.) and No. DMR-8702002 (K.G.W.); and by a Presidential Young Investigator Award, No. PHY-8858250 (R.J.P.).

[1] I. Tamm, J. Phys. (Moscow) **9**, 449 (1945).

[2] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

[3] An extensive set of references can be found in A. Harindranath, Ohio State University report (to be published).

[4] S. J. Brodsky and G. P. Lepage, in *Perturbative Quantum Chromodynamics,* edited by A. H. Mueller (World Scientific, Singapore, 1989), and references therein; H. C. Pauli and S. J. Brodsky, Phys. Rev. D **32**, 1993 (1985); **32**, 2001 (1985); T. Eller, H. C. Pauli, and S. J. Brodsky, Phys. Rev. D **35**, 1493 (1987); K. Hornbostel, S. J. Brodsky, and H. C. Pauli, Phys. Rev. D **41**, 3814 (1990).

[5] For example, see R. Machleidt, K. Holinde, and Ch. Elster, Phys. Rep. **149**, 1 (1987).

[6] K. G. Wilson, in Proceedings of Lattice '89, Capri, Italy, September 1989 (to be published).

[7] K. G. Wilson, Phys. Rev. **140**, B445 (1965).

[8] For a review, see H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, Evanston, IL, 1955), Vol. II.

[9] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[10] A. Harindranath and J. P. Vary, Phys. Rev. D **37**, 3010 (1988).

[11] A. Harindranath and R. J. Perry, Phys. Rev. D (to be published).

[12] R. J. Perry and A. Harindranath, Ohio State University report (to be published).

[13] A. C. Tang, Ph.D. thesis, Stanford University, 1990 [SLAC Report No. SLAC-0351 (unpublished)].

[14] D. Mustaki, S. Pinsky, J. Shigemitsu, and K. Wilson, Ohio State University report (to be published).