

Quantum Tunneling and Chaos in a Driven Anharmonic Oscillator

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(Received 7 June 1990)

The Husimi distribution is computed for a particle in a double-well potential and an oscillatory driving force. The extended phase space of the classical system contains two disjoint stable tubes of regular orbits, embedded in a chaotic sea. For the quantum system we find coherent oscillatory tunneling between these stability tubes, at a rate many orders of magnitude greater than the rate of ordinary undriven tunneling.

PACS numbers: 05.45.+b, 03.65.-w, 73.40.Gk

Although the role of chaos in classical dynamics is well established, its role in quantum mechanics is not so clear. Is "quantum chaos" to be understood solely as a phenomenon that emerges in the classical limit, or are there intrinsically quantal phenomena in which chaos plays a role? In order to investigate this question, we have studied the model of a particle in a double-well potential and an oscillating external driving force. The classical motion of this system has already been studied.¹ In the absence of driving, a separatrix in phase space separates the low-energy orbits that are confined to one well from the high-energy orbits that cross the hump between the wells. When an oscillatory driving force is applied, the separatrix is replaced by a chaotic layer, whose size depends on the amplitude and frequency of the driving force. The undriven quantum system is a textbook example of a distinctively quantal process—the tunneling from one potential well to the other through a classically impenetrable region. We study the driven quantum system in search of interplay between classical phase-space structures and quantum tunneling.

We have found that coherent tunneling takes place between small isolated classically stable regions of phase space. These regions are bounded by KAM (Kolmogorov-Arnol'd-Moser) surfaces, separating regular from chaotic classical orbits, and are much smaller than the volume of a single potential well. The tunneling rate is very much greater than the rate of ordinary tunneling in the absence of driving. Thus we have a distinctively quantal effect (tunneling) whose character is dominated by the effect of classical phase-space structures.

The Hamiltonian of our model is

$$H = p^2/2M + Bx^4 - Dx^2 + \lambda x \cos(\omega_0 t). \quad (1)$$

We present results for the parameter values $M=1$, $B=0.5$, $D=10$, $\lambda=10$, and $\omega_0=6.07$. For the classical model, these yield a large chaotic zone with two small regular regions embedded in it, as is shown by the stroboscopic plots in Fig. 1(a). There is a regular trajectory confined in the stable island centered at $x \approx -1.5$, $p=0$, and another confined in the island centered at $x \approx 4.15$, $p=0$. The cloud of points belongs to a single chaotic

trajectory.

The stable islands are best viewed as cross sections of tubes in an extended phase space in which time is the third dimension.² Figure 1(b) shows cross sections of the

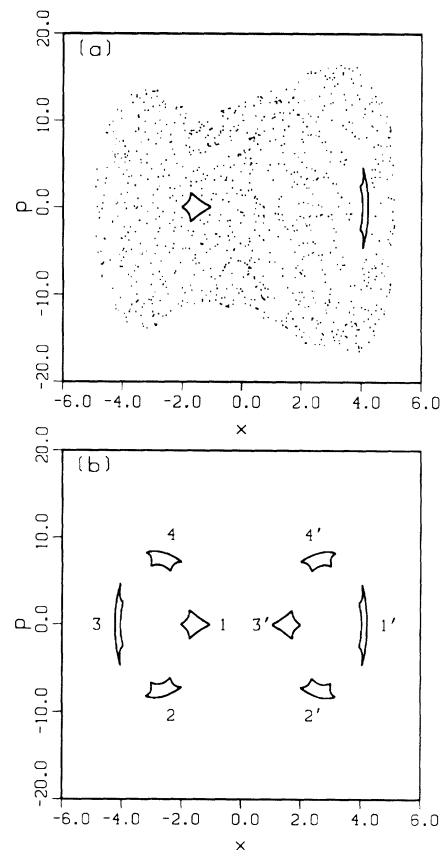


FIG. 1. (a) Stroboscopic plots of three trajectories at $t = n\tau_0$ ($n=0,1,2, \dots, 1000$), where τ_0 is the period of the driving force. One trajectory begins at $x = -3.5$, $p=0$ covers the whole chaotic sea; one begins at $x = -2$, $p=0$ and is confined within the left stable island; one begins at $x = 4.224$, $p=0$ and is confined within the right stable island. (b) Cross sections of the stability tubes in extended phase space at quarter-period intervals. The sections marked 1 and 1' correspond to the stable islands of part (a); sections 2 and 2' are at a time $\tau_0/4$ later; etc.

stable tubes at intervals of $\tau_0/4$, where $\tau_0 = 2\pi/\omega_0$ is the period of the driving force. These “stability tubes” form clockwise spirals through extended phase space, each spiral being centered in one of the potential wells. An orbit that begins in one of the stability tubes remains confined within that tube (and hence in one of the potential wells). An orbit that begins outside of the stability tubes, within the chaotic zone, will cross randomly from one potential well to the other.

The quantum-mechanical Hamiltonian has the form of Eq. (1) with x and p becoming the position and momentum operators. For computational purposes it is convenient to choose as basis vectors the eigenvectors $\{|n\rangle\}$ of a harmonic oscillator, $(\hat{p}^2/2M + \frac{1}{2}M\omega^2\hat{x}^2)|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$. With this basis, the matrix elements of any power of \hat{p} or \hat{x} can easily be evaluated. The frequency ω of the harmonic oscillator is arbitrary, and so may be adjusted to optimize the computation. If it is chosen too large or too small, the number of basis functions required for a given accuracy will increase. In our computation we have used the value $\omega = 6.25$. In this representation, the Schrödinger equation becomes

$$i\hbar \frac{d}{dt} \langle m|\psi(t)\rangle = \sum_{n=0}^{\infty} H_{mn} \langle n|\psi(t)\rangle, \tag{2}$$

with H_{mn} being real and symmetric. For $n \geq m$ it is

$$\begin{aligned} H_{mn} = & \delta_{nm} \{ (n + \frac{1}{2})\hbar\omega + \frac{3}{2}Bg^2[\theta(m-2)m(m-1) + 2m\theta(m-1) + \frac{1}{2}] - \frac{1}{2}D'g(1+2m) \} \\ & + \delta_{n,m+1} \sqrt{g(m+1)/2} \lambda \cos(\omega_0 t) + \delta_{n,m+2} g \sqrt{(m+2)(m+1)} [Bg(m + \frac{1}{2}) - \frac{1}{2}D'] \\ & + \delta_{n,m+4} (g^2/4)B \sqrt{(m+4)(m+3)(m+2)(m+1)}, \end{aligned} \tag{3}$$

where $g = \hbar/M\omega$, $D' = D + \frac{1}{2}M\omega^2$, and $\theta(x) = 1$ for $x \geq 1$ and $\theta(x) = 0$ for $x < 1$.

Rather than computing the wave function $\psi(x,t)$, we compute the Husimi function, defined as

$$\rho(x,p) \equiv (2\pi\hbar)^{-1} |\langle \phi_{x,p}|\psi(t)\rangle|^2, \tag{4}$$

where $|\phi_{x,p}\rangle$ is a minimum-uncertainty function, of Gaussian form in both position and momentum representations, having average position x and average momentum p . $|\phi_{x,p}\rangle$ is a coherent state of a harmonic oscillator, and if we choose it to be the same oscillator whose eigenfunctions form the basis vectors, then we obtain the simple representation

$$\langle \phi_{x,p}|n\rangle = \exp(-\frac{1}{2}|\alpha|^2) \alpha^n / \sqrt{n!}, \tag{5}$$

where $\alpha = (x + igp/\hbar)/\sqrt{2g}$. The Husimi function provides a coarse-grained (on the scale of \hbar) phase-space distribution, and is the closest quantal analog of a classical phase-space distribution. Therefore it is often used to study the effects of classical chaos in quantum systems.³⁻⁷

The results that we shall present are for the value $\hbar = 1$, which, combined with the other parameter values given below Eq. (1), implies that the double-well poten-

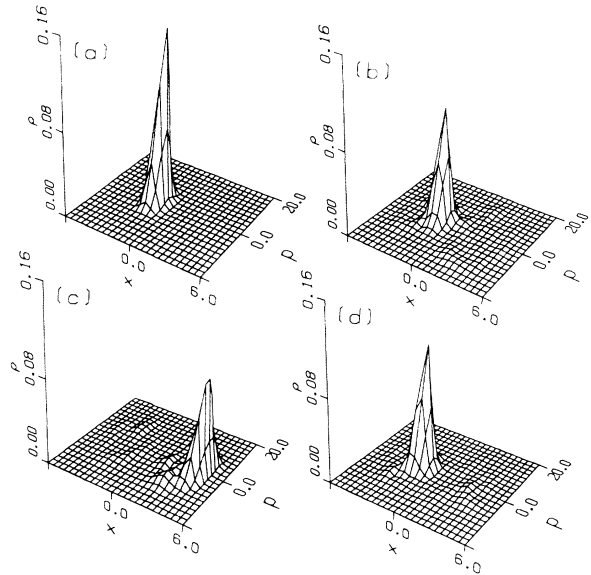


FIG. 2. Time development of the Husimi function of a state with the initial wave packet peaked at $x = -1.5$, $p = 0$. (a) $t = 0$; (b) $t = 5\tau_0$; (c) $t = 115\tau_0$; (d) $t = 230\tau_0$.

tial, $V(x) = Bx^4 - Dx^2$, supports nineteen negative energy levels. The integration of the time-dependent Schrödinger equation (2) used 115 basis vectors. In Fig. 2(a) we chose the initial state to be a minimum-uncertainty Gaussian function centered in the stable island shown on the left-hand side of Fig. 1(a). The other parts of Fig. 2 show the evolution of the Husimi distribution at times $t = 5\tau_0$, $115\tau_0$, and $230\tau_0$. (τ_0 is the period of the driving force.) At $t = 115\tau_0$ [Fig. 2(c)] the particle has (with very large probability) tunneled from the left stability tube to the right stability tube, and at $t = 230\tau_0$ [Fig. 2(d)] the initial state has been approximately restored. We expect these coherent tunneling oscillations will continue for some time (perhaps with some damping due to leakage of probability into the classically chaotic zone).

Figure 3 shows the Husimi distribution at the intermediate times $t = 58\tau_0$, $58.25\tau_0$, $58.5\tau_0$, and $58.75\tau_0$, when the state vector is a superposition of components localized in each of the classical stability tubes. It is apparent from this figure that the major peaks of the Husimi distribution are revolving in the phase plane [compare Fig. 1(b)], corresponding to the helical form of the classical stability tubes in extended phase space.

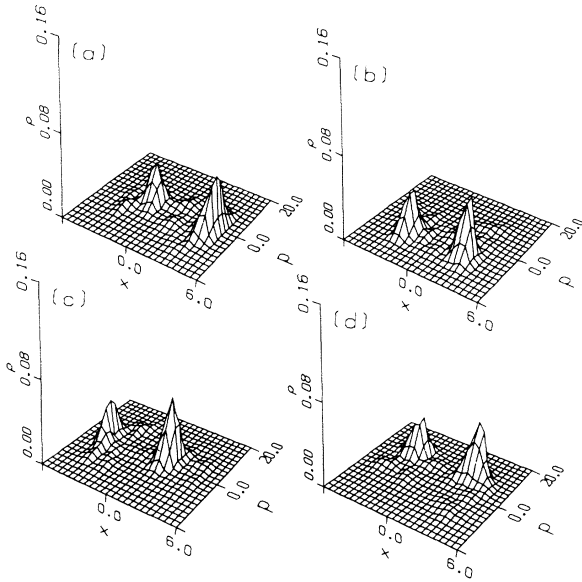


FIG. 3. The same state as in Fig. 2 at (a) $t = 58\tau_0$, (b) $t = 58.25\tau_0$, (c) $t = 58.5\tau_0$, and (d) $t = 58.75\tau_0$.

In Fig. 4 we chose a similar initial state, but located in the classically chaotic zone just outside of the stable island. In contrast to Fig. 2, the Husimi distribution has spread rapidly throughout the chaotic zone already at $t = 5\tau_0$, and it shows no sign of recurring at $115\tau_0$ or $230\tau_0$. Although one cannot rule out quasiperiodic motion for this confined quantum system, the recurrence time must be very much longer than those shown in Fig. 2. Instead of displacing the initial state in phase space at $t = 0$, one can keep the same initial state of Fig. 2(a) but launch it one-half period later at $t = \tau_0/2$. It is then in the chaotic zone, since the stable island has moved from position 1 to position 3 in Fig. 1(b). The result is very similar to that shown in Fig. 4. The character of the time evolution of the Husimi distribution depends primarily on whether it begins within a classical stability tube in extended phase space, or in the classical chaotic zone.

What is the significance of this tunneling phenomenon? After all, tunneling between the two sides of a double-well potential is to be expected. We believe it is significant for several reasons. It has previously been observed that quantum states may be (approximately) localized within regular regions. We find, however, that although a quantum state within a regular region is effectively isolated from the neighboring chaotic zone, it can couple to the interior of another distant, classically inaccessible, regular region. Second, the tunneling rate is many orders of magnitude faster than that of ordinary tunneling. In the absence of the driving force, the tunneling rate is too slow to be detected by our computational technique. A rough but conservative estimate indicates the ordinary (undriven) tunneling time to be

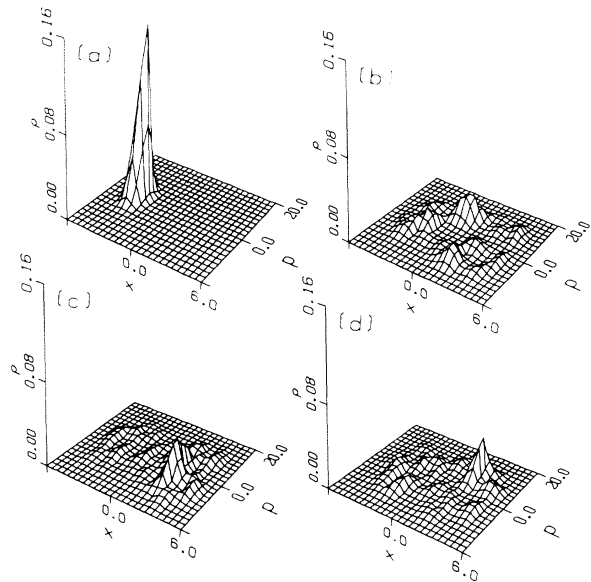


FIG. 4. Time development of the Husimi function of a state with the initial wave packet centered at $x = -3.5$, $p = 0$. (a) $t = 0$; (b) $t = 5\tau_0$; (c) $t = 115\tau_0$; (d) $t = 230\tau_0$.

greater than $10^6\tau_0$, compared to $115\tau_0$ with the driving force. Third, and most surprising, is the coherent oscillatory character of the tunneling. It has already been shown that stable classical KAM surfaces act as permeable barriers to quantum diffusion.^{8,9} Therefore one might expect that the wave packet, initially launched within a stability tube, would gradually leak out into the chaotic zone and become disordered. But, in fact, it oscillates back and forth between the two stability tubes, with the coherence being maintained in spite of the intervening chaotic zone. The robustness of this quantum coherence is quite remarkable. Finally, this phenomenon clearly involves a nontrivial interplay between a purely quantum phenomenon (tunneling) and classical chaos.

This novel tunneling phenomenon is, in principle, accessible to experiment. It bears an obvious similarity to macroscopic quantum coherence (MQC),¹⁰ which is the predicted coherent tunneling oscillation of the magnetic flux in a Josephson junction. The necessary measurement is difficult, but appears to be within the realm of feasibility.¹¹ Our results can be interpreted as predicting a major modification and enhancement of the MQC oscillations due to the interplay between tunneling and classical chaos. Quantum tunneling is also being utilized in electronic devices.¹² Recent experiments¹³ show that the tunneling rate of an electron through a semiconductor double-quantum-well structure can be enhanced by means of a dc bias (resonant tunneling). Our results suggest that the tunneling rate may also be enhanced by means of an ac field or an intense laser beam.

This research is supported by a grant from the Natu-

ral Sciences and Engineering Research Council of Canada.

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