Scattering Theory of Thermal and Excess Noise in Open Conductors

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(Received 28 August 1990)

Thermal fluctuations at equilibrium and excess fluctuations in the presence of transport in open multiprobe conductors are related to the scattering matrix of the conductor. The fluctuation-dissipation theorem for multiprobe conductors is discussed. A general expression for the excess noise in the presence of transport is derived. These results are applied to conductors which exhibit the quantized Hall effect. If backscattering is suppressed, excess noise is also suppressed.

PACS numbers: 72.10.Bg, 73.50.Fq, 73.50.Td

Many novel phenomena in conductors with dimensions small compared to an equilibration length have been successfully analyzed with the help of the transmission approach. This approach[1-3] views the sample as a target at which incident carriers are reflected or transmitted into other probes. The measured electric resistances are related to the scattering matrix of the sample.[3] I supplement this approach with a discussion of the thermal noise properties at equilibrium and of excess noise in the presence of current transport. As shown earlier, the Onsager-Casimir symmetries of the transport coefficients can be related directly to the microreversibility property of the scattering matrix.[3] Under the same assumptions a fluctuation-dissipation theorem must also hold. Since the transmission approach assumes that scattering inside the sample is only elastic, fluctuations must originate solely from thermal agitation in the contacts of the sample. An instructive discussion of the Johnson-Nyquist noise[4] and transmission in a two-terminal conductor has been given by Landauer.[5] Below I present the fluctuation-dissipation theorem for multiprobe conductors. The equilibrium noise, by virtue of the fluctuation-dissipation theorem, does not give any information which cannot be obtained by a resistance measurement. The noise in the presence of transport is more interesting. I show that the excess noise (shot noise) is determined by transport coefficients which are not obtained from a resistance measurement. My discussion of excess noise generalizes results by Lesovik[6] and Yurke and Kochanski.[7] These authors considered simple two-terminal model problems without interchannel scattering. I derive an expression for the excess noise for multiterminal conductors subject to a magnetic field which is valid for an arbitrary scattering matrix. To illustrate these results I discuss specifically the noise properties of conductors (see Fig. 1) which exhibit the quantized Hall effect.[8,9]

I will now describe some of the elements of the calculation. Figure 1 taken from Ref. 10 shows conductors with a number of leads which in turn are connected to electron reservoirs. The final results are independent of the detailed properties of the states in the reservoirs. For simplicity, we characterize each channel by a dispersion $E_{\alpha n}(k) = E_{\alpha n}(0) + \hbar^2 k^2/2m$. Here $E_{\alpha n}(0)$ is the threshold for conduction of the nth channel determined by the lateral confinement of the reservoir. The transverse eigenfunction with eigenvalue $E_{\alpha n}(0)$ is denoted by $\phi_{\alpha n}$. The kinetic energy associated with longitudinal motion is $\hbar^2 k^2/2m$. States with positive velocity are taken to be incident on the conductor. Transport is described by the

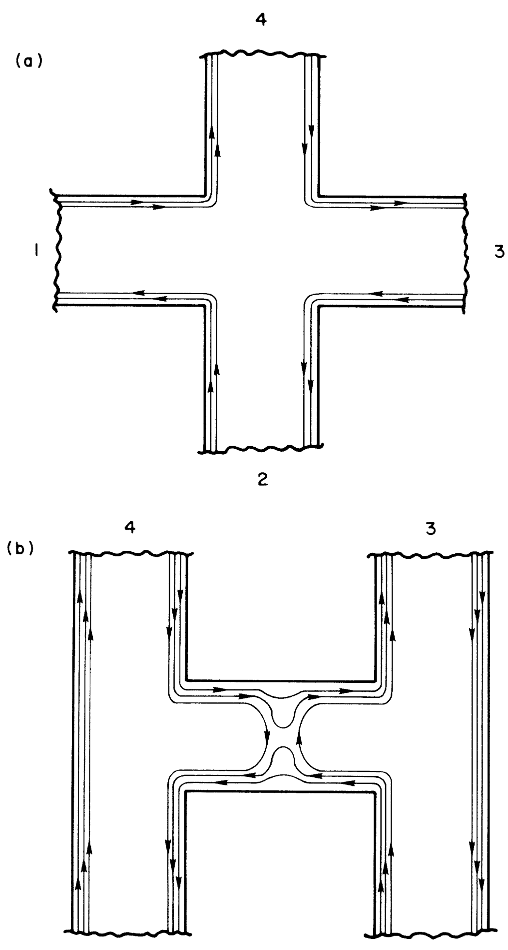

FIG. 1. (a) Conductor in a quantizing magnetic field. (b) Conductor with a gate or constriction. Faint lines depict the edge states at the Fermi energy.

operator $\hat{\Psi} = \sum_{\alpha m} \hat{\Psi}_{\alpha m}$ where

$$\hat{\Psi}_{\alpha m} = \frac{1}{\sqrt{2\pi}} \int dk\, \psi_{\alpha m}(k) \hat{a}_{\alpha m}(k) e^{-i\omega_{\alpha m} t} . \qquad (1)$$

The $\psi_{\alpha m}$ are the scattering states. $a^{\dagger}_{\alpha m}(k)$ annihilates an electron with energy $E_{\alpha n}(k)$. The time evolution is governed by $\hbar\omega_{\alpha m} \equiv E_{\alpha m}(k) - \mu_\alpha$, where μ_α is the chemical potential of reservoir α. To proceed we must evaluate the current operator $\hat{I}_\alpha$ in each reservoir. To this end we need the current matrix elements in lead α which arise if a wave is incident in channel m in lead β and a wave is incident in channel n in lead γ,

$$I_{\beta\gamma mn}(\alpha) \equiv \frac{\hbar}{2mi} \int dy_\alpha \left[\psi^{\dagger}_{\beta m}(\alpha) \frac{d\psi_{\gamma n}(\alpha)}{dx} - \frac{d\psi^{\dagger}_{\beta m}(\alpha)}{dx} \psi_{\gamma n}(\alpha) \right] . \qquad (2)$$

Here dy_α denotes an integration over the cross section of reservoir α. The elastic-scattering properties of the conductor are determined by scattering matrices $s_{\beta\alpha}$ which connect the incident amplitudes in lead α to the outgoing amplitudes in lead β. The asymptotic amplitudes of the scattering state $\psi_{\beta m}$ in probe β consist of an incident wave in channel m, $e^{ik_{\beta m}x}\phi_{\beta m}$, of reflected waves in each channel βm, $(v_{\beta m}/v_{\beta n})^{1/2} s_{\beta\beta nm} e^{-ik_{\beta n}x}\phi_{\beta n}$, and of transmitted waves in each channel γn of all the other probes, $(v_{\beta m}/v_{\gamma n})^{1/2} s_{\gamma\beta nm} e^{-ik_{\gamma n}x}\phi_{\gamma n}$. If both states in Eq. (2) are evaluated at the Fermi energy, I find that $A_{\beta\gamma mn}(\alpha) \equiv (v_{\beta m})^{-1/2}(v_{\gamma n})^{-1/2} I_{\beta\gamma mn}(\alpha)$ can be directly expressed with the help of the scattering matrices $s_{\alpha\beta}$,

$$A_{\beta\gamma}(\alpha) = \delta_{\alpha\beta}\delta_{\alpha\gamma} 1_\alpha - s^{\dagger}_{\alpha\beta} s_{\alpha\gamma} . \qquad (3)$$

The current operator for probe α is

$$\hat{I}_\alpha(t) = \frac{1}{2\pi} \sum_{\beta\gamma mn} \int dk_{\beta m} \int dk_{\gamma n}\, I_{\beta\gamma mn}(\alpha) e^{i(\omega_{\beta m} - \omega_{\gamma n})t} \times a^{\dagger}_{\beta m}(k_{\beta m}) a_{\gamma n}(k_{\gamma n}) . \qquad (4)$$

Equation (4) shows that the current fluctuations are determined by the quantum statistical properties[11] of $\hat{a}^{\dagger}$ and $\hat{a}$ which characterize the occupation of the quantum channels in the reservoirs. Since the average of the occupation probability of each quantum channel depends only on the reservoir index but not on the channel index, the average current depends only on the trace of $A_{\beta\beta}$. The noise of a fluctuating quantity is characterized by the spectral density $\langle I_\alpha I_\beta \rangle_\omega$. The spectral density can be obtained by taking the Fourier transform of the current $\hat{I}_\alpha(\omega) = \int dt\, e^{i\omega t} \hat{I}_\alpha(t)$ and by evaluating $\frac{1}{2}\langle \hat{I}_\alpha(\omega)\hat{I}_\beta(\omega') + \hat{I}_\beta(\omega')\hat{I}_\alpha(\omega)\rangle$ which is equal to $2\pi\langle I_\alpha I_\beta\rangle_\omega \delta(\omega+\omega')$. In the "white-noise limit" which is of interest here, the spectral density is independent of ω and is determined by the current matrix elements at the Fermi energy. Since, on the average, the mean-square fluctuations in the occupation probabilities again depend only on the reservoir index but not on the channel index, I find that the spectral densities are determined by traces of sums of products $A_{\gamma\delta}(\alpha) A_{\delta\gamma}(\beta)$. All density-of-states factors (velocities) cancel and the spectral density depends on the scattering matrix only.

Before discussing the current fluctuations we restate the result for the average currents. The average currents can be expressed with the help of $R_{\alpha\alpha} = \mathrm{Tr}(r^{\dagger}_{\alpha\alpha} r_{\alpha\alpha})$, the total reflection probability in lead α, and $T_{\alpha\beta} = \mathrm{Tr}(t^{\dagger}_{\alpha\beta} \times t_{\alpha\beta})$, the total transmission probability for carriers incident in probe β reaching probe α. The average current in probe α is[3]

$$\langle I_\alpha \rangle = (e/h) \int dE(-df/dE) \times \left[(M_\alpha - R_{\alpha\alpha})\mu_\alpha - \sum_\beta T_{\alpha\beta}\mu_\beta \right] . \qquad (5)$$

In Eq. (5) f is the equilibrium Fermi function. M_α is the number of channels in reservoir α. Equation (5) can be used to calculate the resistances $\mathcal{R}_{\alpha\beta,\gamma\delta} \equiv (V_\gamma - V_\delta)/I$. Here the first pair of indices denotes the current source and sink and the second pair of indices denotes the probes which are used to measure voltages. $\mathcal{R}$ is a four-probe resistance if all indices differ from one another. It is a two-probe resistance if the first and second pair of indices are identical. These resistances[3] obey the reciprocity symmetry $\mathcal{R}_{\alpha\beta,\gamma\delta}(B) = \mathcal{R}_{\gamma\delta,\alpha\beta}(-B)$.

We are now prepared to state the fluctuation-dissipation theorem. The calculation outlined above gives for the mean-square current in the frequency interval $\Delta\nu$ at probe α,

$$\langle (I_\alpha)^2 \rangle = 4\Delta\nu kT(e^2/h) \int dE(-df/dE) \left[\sum_{\beta(\neq\alpha)} T_{\alpha\beta} \right] . \qquad (6)$$

It is determined by the sum of all transmission probabilities which permit carriers to enter probe α from any other probe. Equation (6) for the case of a two-terminal conductor reduces to the Johnson-Nyquist noise formula[4,5] $\langle I^2 \rangle = 4\Delta\nu kTG$, where $G = (e^2/h)\int dE(-df/dE)T$, and $T = \mathrm{Tr}(t^{\dagger}t)$. The currents at differing terminals are in general correlated. I find

$$\langle I_\alpha I_\beta \rangle = -2\Delta\nu kT(e^2/h) \int dE(-df/dE)(T_{\alpha\beta} + T_{\beta\alpha}) . \qquad (7)$$

The correlations between the fluctuating currents at differing probes are determined by the transmission probabilities which link the two probes. If we compare Eqs. (6) and (7) with Eq. (5) and take into account that $\sum_{\beta(\neq\alpha)} T_{\alpha\beta} = M_\alpha - R_{\alpha\alpha}$, we see that the current fluctuations are related to the symmetrized transport coefficients. Equations (6) and (7) are, therefore, a manifestation of the fluctuation-dissipation theorem.

Now I consider the fluctuations in the chemical potentials which counterbalance the current fluctuations. These voltage fluctuations can be found by using Eq. (5) for the fluctuations instead of the averaged quantities. The mean-square voltage difference measured between any pair of leads is given by

$$\langle (V_\alpha - V_\beta)^2 \rangle = 4\Delta\nu kT \mathcal{R}_{\alpha\beta,\alpha\beta}\,, \qquad (8)$$

where $\mathcal{R}_{\alpha\beta,\alpha\beta}$ is a two-terminal resistance. The correlation between voltage differences measured across two pairs of leads is given by

$$\langle (V_\alpha - V_\beta)(V_\gamma - V_\delta) \rangle = 2\Delta\nu kT (\mathcal{R}_{\alpha\beta,\gamma\delta} + \mathcal{R}_{\gamma\delta,\alpha\beta})\,. \qquad (9)$$

If all indices in Eq. (9) differ, the correlation is determined by a symmetrized four-terminal resistance. If two indices coincide, the correlation is determined by a symmetrized three-terminal resistance. For $\alpha=\gamma$ and $\beta=\delta$, Eq. (9) reduces to Eq. (8).

Let us now discuss the thermal fluctuations for the conductors shown in Fig. 1. The field is such that the Fermi energy is between the Nth and $(N+1)$th bulk Landau level. Each bulk Landau level which is below the Fermi energy gives rise to an edge state which intercepts the Fermi energy near the boundary of the sample. The transmission probabilities at the Fermi energy are determined by unidirectional motion of carriers along these edge states, and for the conductor of Fig. 1(a) are given by $T_{41}=T_{34}=T_{23}=T_{12}=N$. All other transmission probabilities are zero.[9] Therefore, the mean-square current fluctuations, Eq. (5), are proportional to N. The correlations between currents at differing probes, Eq. (7), are nonzero only if the probes are directly connected by an edge state. The mean-square voltage fluctuations, Eq. (8), are proportional to N. The correlations between voltage fluctuations measured across different pairs of probes are zero: The longitudinal four-probe resistances vanish and the four-probe Hall resistances are antisymmetric under field reversal. Recent noise measurements by Kil *et al.*[12] do indeed show that the mean-square voltage fluctuations exhibit a white-noise shoulder, whereas correlations of voltage fluctuations do not exhibit a white-noise shoulder. Consider now the conductor in Fig. 1(b). A gate or constriction produces a barrier for electron motion.[13,14] Suppose that K edge states are completely reflected at this barrier. Carriers of one edge state are partially reflected with probability R and transmitted with probability T. Thus the total probability for reflection at the gate is $K+R$. The transmission probabilities which determine the mean-square current fluctuations according to Eq. (6) always add up to N independent of the applied gate voltage. Some of the current correlations, Eq. (7), are sensitive to whether or not a barrier is present. For instance, $\langle I_2 I_3\rangle$ is proportional to $K+R$ but $\langle I_1 I_3\rangle=0$ independent of gate voltage. The mean-square voltage fluctuations are also independent of the applied gate voltage. Since the Hall resistance is no longer antisymmetric in this conductor and since the longitudinal resistance is no longer zero, some of the correlations now depend on the gate voltage and are proportional to $(K+R)/(N-K-R)$.

Next I investigate fluctuations away from the average current in the presence of transport. Transport causes fluctuations in excess of the equilibrium noise. A classical discussion predicts shot noise $\langle(\Delta I)^2\rangle = 2e\Delta\nu I$ due to the uncorrelated transfer of carriers through the sample. Using Eq. (3) for a two-terminal conductor, I find in the zero-temperature limit,

$$\langle (\Delta I)^2 \rangle = 2(e^2/h)\Delta\nu |eV| \mathrm{Tr}(r_{11}^\dagger r_{11} t_{12}^\dagger t_{12})\,. \qquad (10)$$

Equation (10) is valid for arbitrary elastic scattering and is valid in the presence of a magnetic field. Alternatively, we could express this result with the help of the transmission matrix only. Since $r_{11}^\dagger r_{11} + t_{12}^\dagger t_{12} = 1$, it follows that the trace in Eq. (10) can also be expressed as $\mathrm{Tr}(t_{12}^\dagger t_{12}(1 - t_{12}^\dagger t_{12}))$. Denote the eigenvalues of the matrix $t_{12}^\dagger t_{12}$ by $T_n(B)$ and the eigenvalues of the matrix $r_{11}^\dagger r_{11}$ by $R_n(B)$. Since $R_n(B)+T_n(B)=1$ and $R_n(B) = R_n(-B)$, the eigenvalues $T_n(B)$ are also symmetric functions of the magnetic field. Equation (10) for the excess noise takes the form

$$\langle (\Delta I)^2 \rangle = 2(e^2/h)\Delta\nu |eV| \sum T_n (1 - T_n)\,. \qquad (11)$$

This is the result given by Lesovik.[6] If all the eigenvalues of the transmission matrix are small compared to one, Eq. (11) reduces to the standard expression for shot noise $\langle(\Delta I)^2\rangle = 2e\Delta\nu I$. Most interestingly, Eqs. (10) and (11) tell us that the shot noise will be smaller than expected from the standard result whenever the transmission matrix has one or more eigenvalues equal or comparable to 1. This result suggests that at low enough temperatures electron motion through open quantum channels is correlated. Open quantum channels occur in resonant double barriers, in split-gate constrictions, and in systems subject to quantizing magnetic fields. Experiments by Li *et al.*[15] show a clear suppression of shot noise in double barriers and indicate that shot noise is suppressed in split-gate constrictions.

Next I present the general result for shot noise in multiprobe conductors. At $kT=0$ I find

$$\langle \Delta I_\alpha \Delta I_\beta \rangle = 2\Delta\nu (e^2/h) \sum_{\gamma,\delta(\gamma\neq\delta)} \int dE\, f_\gamma (1 - f_\delta) \times \mathrm{Tr}(s_{\alpha\gamma}^\dagger s_{\alpha\delta} s_{\beta\delta}^\dagger s_{\beta\gamma})\,. \qquad (12)$$

The mean-square current fluctuations are real and positive: If $\alpha=\beta$ in Eq. (12), each matrix s occurs together with its Hermitian conjugate. If we consider the correlations between excess currents, the coefficients in Eq. (12) are not real. However, the sum of all terms is real and negative on account of the unitary relation $\sum_\delta s_{\alpha\delta} s_{\beta\delta}^\dagger = 0$. Again we calculate the excess voltage fluctuations by using Eq. (5) for the fluctuating currents and voltages instead of the averaged quantities. I have not found a closed expression analogous to Eq. (9) for the excess voltage fluctuations.

Let us now return to our examples. The conductor of Fig. 1(a) exhibits *no* excess noise. All nonzero transmission probabilities are quantized. The conductor of Fig. 1(b) also exhibits no excess noise as long as transmission through the barrier created by the gate is quantized. The mean-square currents and the correlations given by Eq. (12) are zero. For the situations shown in Fig. 1 each quantum channel is fed by only one reservoir. (Elastic inter-edge-state scattering from one edge state to another away from the gate does not change this result.) In the conductor of Fig. 1(b) excess noise arises only in a transition region where at least one of the edge states is only partially reflected or transmitted. Consider the case where current flow is from contact 1 to contact 2. The mean-square current fluctuations are nonzero only at terminal 1 and terminal 3. I find that $\langle(\Delta I_1)^2\rangle$ and $\langle(\Delta I_3)^2\rangle$ are both proportional to $2\Delta\nu(e^2/h)RT(\mu_1-\mu_2)$. Only the correlation between the currents at terminals 1 and 3 is nonzero and given by $-2\Delta\nu(e^2/h)\times RT(\mu_1-\mu_2)$. The resulting mean-square voltage fluctuations measured between terminals 1 and 4 and measured between terminals 2 and 3 are zero. The mean-square voltage fluctuations between any other pair of terminals are given by

$$2\Delta\nu(h/e)[RT/(N-K-R)^2](V_1-V_2)\,. \qquad (13)$$

Equation (13) predicts that the excess voltage fluctuations become larger the more edge states are reflected. If all but one edge state are completely reflected, $K=N-1$, Eq. (13) is proportional to R/T and shows that the excess voltage fluctuations diverge as the conduction path is pinched off. Similar results are obtained if the current source and sink are at different terminals than discussed above.

Observation of the absence of excess noise in situations where there is no backscattering would further enhance our understanding of transport in the quantum Hall regime. Different theoretical models of the fractional quantum Hall effect[16] likely give differing results for the noise properties and it would also be interesting to compare these with experiments.

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