Theory of Dissipative Trapped-Ion Convective-Cell Turbulence

P. H. Diamond and H. Biglari

Department of Physics, B-019, University of California at San Diego, La Jolla, California 92093 and General Atomics, San Diego, California 92138

(Received 15 June 1990)

The nonlinear dynamics of dissipative trapped-ion convective cells in broadband turbulence is analyzed. Saturation is achieved by both local *and* nonlocal transfer of spectral energy from unstable long-wavelength trapped-ion fluctuations to shear-damped moderate-wavelength trapped-electron modes. In particular, contrary to previous studies, it is shown conclusively that no long-wavelength condensation of fluctuation energy is possible. The transport level associated with these structures is therefore moderate and not catastrophic (Bohm-like), as conventionally believed.

PACS numbers: 52.35.Ra, 52.25.Gj, 52.35.Qz

Although to date there exists no experimental identification of fluctuations at very long wavelengths (i.e., $k_\theta \rho_i \lesssim 0.2$, where $\rho_i = v_{ti}/\Omega_i$ is the ion Larmor radius, v_{ti} is the ion thermal speed, and $\Omega_i = eB/m_i c$ the ion gyrofrequency), it is a common feature of scattering measurements in high-temperature plasma discharges that the observed spectra increase monotonically towards long wavelengths without evincing a peak. This concentration of fluctuation energy at long wavelengths underscores the need to assess theoretically the impact of long-wavelength turbulence on plasma confinement and transport. The most promising candidates in this realm are trapped-ion convective cells[1] (TICC's) which, due to their rapid bounce motion, average out any stabilizing influence associated with shear damping and thereby grow to large scales. Previous work on transport associated with these instabilities predicts an extremely unfavorable scaling of the diffusion coefficient with temperature[2] (i.e., $D \propto T^{21/2}$) near marginal stability, and catastrophic, Bohm-like transport concentrated at long wavelengths.[3] Neither of these predictions have been borne out by experiments which operate in regimes where such turbulence is expected to be operative. In this Letter, we reexamine the nonlinear theory of TICC's and show that, in fact, no long-wavelength condensation[3] of fluctuation energy is possible. Although TICC's can grow to large fluctuation amplitudes, they have slow radial motion, so that the ensuing transport level is moderate and comparable to that associated with trapped-electron turbulence. We will focus on the so-called dissipative trapped-ion convective cells (DTICC's) for simplicity and ease of comparison with previous work, although similar conclusions also follow for other branches of the TICC family of instabilities and will be presented elsewhere.

Dissipative trapped-ion convective cells[1] are dominantly nondispersive modes which propagate in the electron diamagnetic direction, are destabilized by inverse trapped-electron collisional dissipation at intermediate wavelengths, and are stabilized by ion collisions at very long wavelengths. The linear dispersion relation for these modes is given by

$$\omega = \frac{\epsilon^{1/2}}{2}\omega_{*e}\left(1 - \frac{k_\perp^2\rho_{bi}^2}{2}\right) + i\left[\frac{\epsilon}{4}\frac{\omega_{*e}^2}{\nu_{\mathrm{eff},e}}(1 - k_\perp^2\rho_{bi}^2) - \nu_{\mathrm{eff},i} - \frac{\epsilon^{1/2}\pi^{1/2}\omega_{*e}}{4}\left(\frac{\omega_{*e}}{\omega_{ti}}\right)^3\left(1 - \frac{3\eta_i}{2}\right)\right], \tag{1}$$

where $\epsilon = r/R$ is the inverse aspect ratio, $\omega_{*j} = k_\theta \rho_j v_{tj}/L_n$ is the diamagnetic drift frequency, $\omega_{ti} = v_{ti}/qR$ is the circulating ion transit frequency, $\nu_{\mathrm{eff},j} = \nu_j/\epsilon$ is the effective collision frequency of the jth species, $\rho_{bi} = (2/\epsilon)^{1/2} q\rho_i$ is the trapped-ion banana width, $\eta_i = d\ln T_i/d\ln n_i$, $L_n^{-1} = d\ln n_i/dr$, and T_e has been set equal to T_i for simplicity. Finite banana-width effects introduce finite dispersion and damp the mode at short wavelengths, while transiting ions give rise to ion Landau damping (growth) for η_i less (greater) than $\frac{2}{3}$. Spatially, the mode maximizes itself halfway between adjacent rational surfaces radially and balloons to the unfavorable-curvature side poloidally. The radial and poloidal scales are coupled together as a consequence of the radial phase shift between the helicity of the field (tracked by the trapped ions) and that of the mode, i.e., $k_r = k_\theta \hat{s}$.

The nonlinear dynamics of DTICC's can be described by a simple one-field, quasi-two-dimensional fluid model, characterized by two qualitatively different types of nonlinearities, a one-dimensional shocklike nonlinearity,[2,4] and a two-dimensional $\mathbf{E}\times\mathbf{B}$ advective nonlinearity:[1]

$$\frac{\partial \tilde{n}}{\partial t} + \frac{v_{*i}^T}{2}\frac{\partial \tilde{n}}{\partial y} + \nu_{\mathrm{eff},i}\tilde{n} + \frac{(v_{*i}^T)^2}{4\nu_{\mathrm{eff},e}}\frac{\partial^2 \tilde{n}}{\partial y^2} + \frac{v_{*i}^T}{\epsilon^{1/2}}(1-\eta)\frac{\partial \tilde{n}^2}{\partial y} - \frac{L_n (v_{*i}^T)^2}{\epsilon^{1/2}\nu_{\mathrm{eff},e}}\hat{\mathbf{e}}_\parallel \cdot \nabla \frac{\partial \tilde{n}}{\partial y} \times \nabla \tilde{n} = 0\,, \tag{2}$$

where $\tilde{n} = \epsilon^{1/2}\tilde{n}^{\mathrm{tr}}/n$ is the normalized trapped-ion density field, $y=r_{mn}\theta$ is the poloidal variable, $v_{*i}^{T}=\epsilon^{1/2}\omega_{*i}/k_\theta$, and $\hat{\mathbf{e}}_\parallel$ is the unit vector in the direction of the magnetic field. When the advective nonlinearity is ignored, the nonlinear equation becomes one dimensional, and forms the basis of the nonlinear model analyzed in Refs. 2 and 4 near marginal stability.[5] However, we note here that the neglect of the advective nonlinearity undermines the utility of the equation as a faithful model of trapped-ion turbulence in tokamaks in two crucial respects. First, given that the ratio of the advective to the shock nonlinearity is $k_r L_n \omega_{*e}^T/\nu_{\text{eff},e} \propto T^{5/2}$, the former rapidly becomes dominant with increasing temperature. Second and more importantly, the added degree of freedom introduced by two dimensionality allows for a more efficient mechanism for the system to transfer unstable fluctuation energy to its sink where it can be dissipated. Stated differently, a steady-state configuration is likely to be achieved more efficiently, i.e., at *lower* transport levels, when the system can relax in two dimensions than in one. Finally, for any realistic set of parameters, there exists a *broad* range of unstable wave numbers, so that the assumption of proximity to marginal stability,[2,4] while interesting from the point of view of transition from laminar to turbulent behavior, is unlikely to capture the situation encountered in practice. The case at hand is quite distinct from that of, for example, ion-temperature gradient-driven turbulence, where the instability threshold associated with η_i can enforce proximity to marginality.[6] With these considerations in mind, we are led to ignore the shock nonlinearity, and focus on the nonlinear dynamics of the leftover equation, which we will refer to as the Kadomtsev-Pogutse equation (KPE), in *broadband* turbulence.

Critical issues in the theory of DTICC turbulence include both the direction of and mechanism for spectral transfer. The former is especially crucial to any concept of a saturated state of DTICC turbulence, since the available dissipation at large scales is exceedingly weak. Insight into the direction of spectral flow may be gleaned from consideration of the equilibrium statistical mechanics of the KPE in the limit of vanishing growth and damping. It should be remarked that neglect of linear growth and damping is *not* equivalent to ignoring dissipation here, as the nonlinearity, albeit conservative, depends explicitly on the collision frequency. This model supports a single, nontrivial quadratic integral invariant, which corresponds to the total fluctuation amplitude: $\int d^2 r\, \tilde{n}^2$. Following standard procedure,[7] it follows that the canonical probability distribution of equilibrium states is $P=Z^{-1}\exp(-\alpha\sum_k|\tilde{n}_k|^2/2)$, and that the equilibrium spectrum for the KPE is $E_n(k)=\pi k|\tilde{n}_k|^2=\pi k/\alpha$, which corresponds to equipartition. Here, α is the effective temperature, and

$$Z=\int\cdots\int\prod_k d\tilde{n}_k \exp\left(-\alpha\sum_k|\tilde{n}_k|^2/2\right)$$

is the partition function. As the two-dimensional system is ultraviolet divergent, the transfer of fluctuation energy is towards *small* scales. In spite of being quasi-two-dimensional, this conclusion is in distinct contrast to the Hasegawa-Mima equation,[8] which is a well-known paradigm for *short*-wavelength drift-wave turbulence. In that case, the constraint of conservation of energy and enstrophy results in the short-wavelength *decay* of the energy (but not enstrophy) spectrum, resulting in a dual cascade scenario. This observation supports the concept that it is the dynamical constraints, rather than the number of spatial dimensions, which determine the direction of spectral transfer. Thus, in contrast to conventional belief,[3] no long-wavelength condensation of fluctuation energy is possible within the context of the KP model of DTICC turbulence.

The second critical issue concerns the mechanism of nonlinear interaction. Three possibilities exist,[9] namely, (nonlinear) wave-wave interaction, wave-particle interaction, and eddy interaction (strong turbulence mode coupling). As DTICC frequencies are nondispersive, wave-wave interactions are irrelevant as all triads are resonant (i.e., if $\mathbf{p}+\mathbf{q}=\mathbf{k}$, then $\omega_p+\omega_q=\omega_k$). Similarly, for wave-particle interactions, which generically yield a nonlinear transfer rate of the form

$$T(\mathbf{k})=\sum_{\mathbf{k}'} C(\mathbf{k},\mathbf{k}')g_{k,k',k''}|\tilde{n}_{k'}|^2(\chi_k-\chi_{k'}),$$

the observation that in the absence of dispersion the trapped-ion susceptibility χ_k is a constant indicates that $T(\mathbf{k})\to 0$. Here, $C(\mathbf{k},\mathbf{k}')$ is a coupling coefficient, and $g_{k,k',k''}$ is a measure of the wave-particle decorrelation time. Thus, processes such as Compton and induced scattering vanish identically, to all orders in perturbation theory. Hence, eddy interaction is the dominant process in DTICC turbulence. Moreover, the absence of a three-mode frequency mismatch due to dispersion suggests that DTICC's are always in a state of "strong" turbulence, even if fluctuation levels are infinitesimal.

The equation for DTICC spectral evolution due to eddy interaction may be obtained using standard closure methods.[10] We obtain

$$\frac{1}{2}\frac{\partial\langle|\tilde{n}|^2\rangle_k}{\partial t}-\gamma_k\langle|\tilde{n}|^2\rangle_k+T(\mathbf{k})=0\,, \tag{3}$$

where

$$T(\mathbf{k})=\left(\frac{L_n(v_{*i}^T)^2}{\epsilon^{1/2}\nu_{\text{eff},e}}\right)^2\left(\sum_{\mathbf{k}'}|\hat{\mathbf{e}}_\parallel\cdot\mathbf{k}\times\mathbf{k}'|^2k_y'(k_y'-k_y)g_{k,k',k''}\langle|\tilde{n}|^2\rangle_{k'}\langle|\tilde{n}|^2\rangle_k+\sum_{\mathbf{k}'}|\hat{\mathbf{e}}_\parallel\cdot\mathbf{k}\times\mathbf{k}'|^2(k_y'^2-k_y^2)g_{k,k',k''}\langle|\tilde{n}|^2\rangle_{k'}\langle|\tilde{n}|^2\rangle_k\right.$$
$$\left.-\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}}|\hat{\mathbf{e}}_\parallel\cdot\mathbf{p}\times\mathbf{q}|^2p_y(p_y-q_y)g_{k,k',k''}\langle|\tilde{n}|^2\rangle_p\langle|\tilde{n}|^2\rangle_q\right) \tag{4}$$

is the nonlinear transfer rate, and

$$g^{-1}_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = -i(\omega_k+\omega_{k'}-\omega_{k''})+(\Delta\omega_k+\Delta\omega_{k'}+\Delta\omega_{k''})-(\gamma_k+\gamma_{k'}+\gamma_{k''})=(\Delta\omega_k+\Delta\omega_{k'}+\Delta\omega_{k''})-(\gamma_k+\gamma_{k'}+\gamma_{k''})$$

is the memory function for three-mode interactions. Here, $\Delta\omega_k$ is the nonlinear decorrelation frequency of mode $\mathbf{k}$, and $\gamma_k=\epsilon\omega_{*e}^2/4\nu_{\text{eff},e}-\nu_{\text{eff},i}$ is the linear growth/damping rate. The first and third terms in $T(\mathbf{k})$ represent the physical process of advection of the density fluctuation by the velocity field

$$\tilde{\mathbf{v}}_r=[L_n(v^T_{*i})^2/\epsilon^{1/2}\nu_{\text{eff},e}]\hat{\mathbf{e}}_\parallel\times\nabla\partial\tilde{n}/\partial y\,.$$

The resulting spectral transfer to small scales is identical to that of a passive scalar strained by the velocity field above. The self-consistent backreaction of the density fluctuations on the flow is represented by the second term in Eq. (4). Note that this contribution is negative (positive) for k_y^2 greater (lesser) than $k_y'^2$, corresponding to *nonlocal* nonlinear transfer from *large to small* scales.[11] All told, the structure of the closure approximation indicates that energy flows from large to small scales in the KP model, in accordance with the predictions of equilibrium statistical mechanics. It is easily verified that $\sum_{\mathbf{k}}T(\mathbf{k})=0$, so that the closure approximation to the nonlinearity conserves energy. The form of the nonlinear decorrelation rate is suggested by Eq. (3) and thus chosen to be

$$\Delta\omega_k=\left(\frac{L_n(v^T_{*i})^2}{\epsilon^{1/2}v_{\text{eff},e}}\right)^2 \times\sum_{\mathbf{k}'}|\hat{\mathbf{e}}_\parallel\cdot\mathbf{k}\times\mathbf{k}'|^2k_y'^2g_{k,k',k''}\langle|\tilde{n}|^2\rangle_{k'}\,. \qquad (5)$$

Noting that $g_{k,k',k''}$ (and hence by extension, $\Delta\omega_k$) is positive definite, it may easily be shown that

$$\Delta\omega_{\bar{k}}=\begin{cases}\Delta\omega_{\bar{k}}^{(0)} \text{ if } \Delta\omega_{\bar{k}}^{(0)}>\gamma_{\bar{k}}\,,\\ \gamma_{\bar{k}}+2(\Delta\omega_{\bar{k}}^{(0)})^2/\gamma_{\bar{k}} \text{ if } \Delta\omega_{\bar{k}}^{(0)}<\gamma_{\bar{k}}\,,\end{cases}$$

where

$$\Delta\omega_{\bar{k}}^{(0)}=\frac{L_n(v^T_{*i})^2}{\epsilon^{1/2}\nu_{\text{eff},e}}\left(\sum_{\mathbf{k}'}|\hat{\mathbf{e}}_\parallel\cdot\mathbf{k}\times\mathbf{k}'|^2k_y'^2\langle|\tilde{n}|^2\rangle_{k'}\right)^{1/2} \qquad (6)$$

is the statistically approximated eddy straining frequency. Thus $g_{k,k',k''}$ and $\Delta\omega_{\bar{k}}$ are well behaved in both the small- and large-fluctuation-level regimes. The notation $\bar{k}$ is used to indicate that the forms for $\Delta\omega_k$ given above are best viewed as spectrum-averaged decorrelation rates. Finally, since the advecting velocity field scales as $\partial\tilde{n}/\partial y$, coherent structures in the KP model will necessarily be anisotropic, in that $\tilde{n}=\tilde{n}\,\partial\tilde{n}/\partial y$ is required to annihilate nonlinear interaction.

Saturated fluctuation levels may now be estimated using the stationarity condition $\gamma_k+T(\mathbf{k})=0$, with $\gamma_k>0$, and assuming moderate or strong turbulence levels (this assumption is easily verified *a posteriori*). It follows directly that $\tilde{n}^{\text{tr}}/n\simeq(k_rL_n)^{-1}\simeq(k_\theta\hat{s}L_n)^{-1}$. Hence, as DTICC spatial scales are large, the predicted density and potential fluctuation levels are also large, i.e., $e\tilde{\phi}/T_e\simeq\tilde{n}^{\text{tr}}/n\lesssim(v_{*i}/2v_{ti})L_s/L_n$, where $L_s=qR/\hat{s}$ is the shear length. It is worth noting, however, that the fluctuating *velocities* are no larger than those expected for conventional, short-wavelength dissipative trapped-electron mode turbulence, i.e., $|\tilde{v}_r|\simeq c_s\rho_s/L_n\hat{s}$, where $c_s=(2T_e/m_i)^{1/2}$ and $\rho_s=c_s/\Omega_i$. Thus DTICC's are large in amplitude, but have slow radial motion. It should be pointed out that the "mixing-length" result recovered here follows as a direct consequence of balance between growth and nonlinear transfer, rather than from the balance between nonlinear decorrelation and frequency mismatch, as is usually the case for dispersive modes.[8] Finally, the validity of these results is inexorably tied to the demonstration of nonlinear transfer to small scales and the absence of long-wavelength condensation of fluctuation energy.

The anomalous transport caused by DTICC's is now easily determined. The particle (D) and electron thermal (χ_e) diffusivities may be calculated in the quasilinear approximation, due to the fact that the collision rate $\nu_{\text{eff},e}>\omega,\Delta\omega_k$, and so determines the electron-cell decorrelation time. Straightforward application of the previously derived saturated fluctuation levels yields the result

$$\chi_e\simeq\tfrac{3}{2}D\simeq\tfrac{3}{8}\epsilon\rho_i^2v_{ti}^2/\nu_{\text{eff},e}L_n^2\hat{s}^2\,. \qquad (7)$$

The ion thermal diffisivity is given by

$$\chi_i=\sum_{\mathbf{k}}\langle|v_r|^2\rangle_k\Delta\omega_k/\omega_k^2\simeq\chi_e\,,$$

since trapped-ion diffusion is nonresonant. It is thus readily apparent that trapped-ion diffusion is not "catastrophic," as conventionally believed.[3] Indeed, our result, which resembles that originally suggested by Kadomtsev and Pogutse,[1] is comparable to predictions based on standard drift-wave models.[12] This follows from the aforementioned observation that DTICC's are radially broad, but slow. In light of recent results, which indicate that short-wavelength drift-wave turbulence-driven transport is smaller than heretofore believed,[13] it appears that TICC turbulence is, in the final analysis, *the* dominant agent of anomalous transport in the high-temperature core of tokamak plasmas.

In marked contrast to more conventional models, the KP model of DTICC's cannot, and in fact, *need not*, account for the ultimate disposition of the energy that is transferred to small scales. This is a consequence of the fact that the energy transferred to progressively smaller scales will eventually arrive at a $\mathbf{k}^{\max}$ such that $\omega_{k^{\max}}$

$\gtrsim \omega_{bi}$, which falls outside the domain of the KP model, and into the realm of short-wavelength electron drift-wave turbulence. In other words, the "long-wavelength," shear-damped part of the electron drift-wave turbulence spectrum acts as the "dissipation range" for DTICC turbulence. The magnitude of this spectral outflow of energy may be calculated from

$$\Pi_{k\,\max} = \int_0^{\mathbf{k}^{\max}} d\mathbf{k}' T(\mathbf{k}') \simeq \int_0^{\mathbf{k}^{\max}} d\mathbf{k}' \Delta\omega_{k'} \langle |\tilde{n}|^2 \rangle_{k'} ,$$

where $k_y^{\max} = 2\omega_{bi}/v_{*i}^T$ is the maximum wave number for the DTICC spectral range. Using the saturation amplitudes obtained earlier, it follows directly that $\Pi_{k\,\max} \simeq \epsilon D/L_n^2$, where D is given by Eq. (7). Thus, in the vein of Prandtl mixing-length theory for turbulent shear flows,[14] the net spectral flow rate (referred to as the dissipation rate in the case of shear flows) is equal to the rate of mean energy dissipation by turbulent radial transport, which, in turn, equals the inverse confinemen time, τ_P^{-1}; i.e., $\Pi_{k\,\max} \simeq \tau_P^{-1}$. This observation also implies that transfer from long-wavelength DTICC's *is a significant source of fluctuation energy for short-wavelength drift-wave turbulence,* and that theoretical models thereof should be reassessed by including such a long-wavelength source.

Finally, it needs to be pointed out that due to their large cell dimensions, these modes are susceptible to shearing by differential toroidal rotation, such as can be expected, for example, with parallel neutral-beam injection. Suppression of these fluctuations can be expected when the shearing frequency becomes comparable or dominates the nonlinear decorrelation frequency. The shearing frequency, for differential toroidal rotation, is given by

$$\omega_s = k_\zeta V_\zeta(r) = (n/R) V_\zeta' \Delta r \simeq (\epsilon/q\hat{s}) V_\zeta' ,$$

where n is the toroidal mode number, and $V_\zeta' = dV_\zeta/dr$ is the flow shear. The criterion for mode suppression is then given by

$$V_\zeta' \gtrsim v_{ti}^2 / 4\nu_{\mathrm{eff},e} \epsilon q \hat{s} L_n^2 . \qquad (8)$$

In conclusion, we have given the first detailed nonlinear analysis of dissipative trapped-ion convective cells in *broadband* turbulence. We have shown, utilizing both equilibrium statistical-mechanical arguments and semi-quantitative closure studies, that the fluctuation energy can only flow from long to short scales, thus clearing up a long-standing misconception about the character of these modes. Calculations indicate that the ensuing transport associated with these fluctuations is moderate and *not* Bohm-like. Finally, we remark that the unfavorable $T_e^{7/2}$ temperature scaling of the transport coefficients is also not unreasonable, particularly in light of recent *controlled* perturbative transport studies on TFTR.[15]

The authors would like to acknowledge useful discussions with Professor M. N. Rosenbluth. This work was supported by the U.S. Department of Energy Contract No. DE-FG03-85-ER53275. One of us (P.H.D.) would like to acknowledge an Alfred P. Sloan Research Fellowship, and a Presidential Young Investigator award.

[1]B. B. Kadomtsev and O. P. Pogutse, in *Reviews of Plasma Physics,* edited by N. Leontovich (Consultants Bureau, New York, 1970), Vol. 5, p. 387.

[2]B. I. Cohen *et al.*, Nucl. Fusion **16**, 971 (1976).

[3]R. Saison, H. K. Wimmel, and F. Sardei, Plasma Phys. **20**, 1 (1978).

[4]R. E. LaQuey *et al.*, Phys. Rev. Lett. **34**, 391 (1975).

[5]The model equation analyzed in the cited references also includes dispersive effects associated with ion Landau damping in order to counterbalance shock formation. The resulting equation is known as the Kuramoto-Sivashinsky equation in the fluid literature, although it was first derived in Ref. 4.

[6]H. Biglari, P. H. Diamond, and P. W. Terry, Phys. Rev. Lett. **60**, 200 (1988); N. Mattor and P. H. Diamond, Phys. Fluids B **1**, 1980 (1989).

[7]R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).

[8]A. Hasegawa and K. Mima, Phys. Rev. Lett. **39**, 205 (1977).

[9]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969).

[10]S. Orszag, J. Fluid Mech. **41**, 363 (1970).

[11]R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).

[12]P. L. Similon and P. H. Diamond, Phys. Fluids **27**, 916 (1984).

[13]F. Y. Gang, P. H. Diamond, and M. N. Rosenbluth (to be published).

[14]A. A. Townsend, *The Structure of Turbulent Shear Flow* (Cambridge Univ. Press, Cambridge, 1976), p. 135.

[15]P. C. Efthimion *et al.* (to be published).