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Existence of Stable Orbits in the x^2y^2 Potential

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We have found the presence of at least one family of stable periodic orbits in the system $H=\frac{1}{2}(p_x^2+p_y^2+x^2y^2)$. This counterproves earlier claims that the x^2y^2 potential gives fully ergodic motion.

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An active field in the study of Hamiltonian dynamics has been the search for and study of globally chaotic systems, i.e., fully ergodic systems in which all periodic orbits are unstable and which do not exhibit the complicated division of phase space (mixed stability, Kol'mogorov-Arnol'd-Moser tori, elliptic island structure, etc.) otherwise associated with generic Hamiltonian systems. Examples of such systems where global ergodicity has been rigorously proven are Sinai billiards[1,2] and motion on Riemann surfaces of constant negative curvature.[3-6] The question whether there exists some *analytic* Hamiltonian in Euclidean space with this property still remains open. A candidate has for a long time been the following one: $H=\frac{1}{2}(p_x^2+p_y^2+x^2y^2)$. A number of papers have claimed or in various ways argued that this system is ergodic.[7-15] As we will see, this conjecture is not true.

The scaling properties of the Hamiltonian permit us to study the system at one single energy, which we have chosen to be $E=\frac{1}{2}$. Furthermore, the C_{4v} symmetry of the system restricts the effective motion to the fundamental domain $\{(x,y)\colon\ x\geq y\geq 0\}$, where the borderlines $y=0$ and $x=y$ are treated as hard walls. Successive crossings of (or rather bounces on) the Poincaré sur-

TABLE I. Coordinates at $y=0$ for an orbit with period $n=11$.

i	x_i	$p_{x,i}$
1	3.146 401 23	0.001 793 19
2	3.068 281 26	−0.156 894 46
3	2.812 484 87	−0.324 112 47
4	2.301 426 39	−0.514 358 54
5	1.175 498 20	−0.773 406 80
6	0.781 328 80	0.481 317 45
7	0.757 378 63	−0.496 210 17
8	1.130 339 93	0.781 278 02
9	2.285 341 02	0.519 178 81
10	2.804 174 09	0.328 111 66
11	3.064 574 98	0.160 571 14

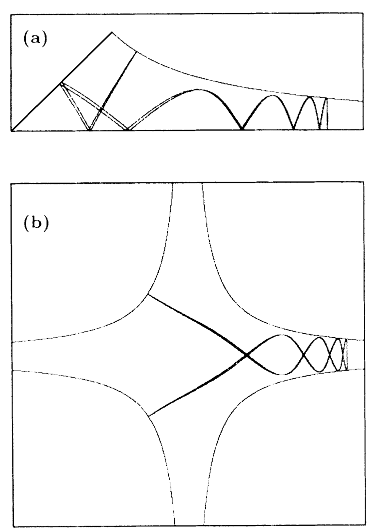

FIG. 1. The trajectory of the orbit of Table I (a) in the fundamental domain and (b) in the full (x,y) plane. The contour $x^2y^2=1$ encloses the energetically allowed region.

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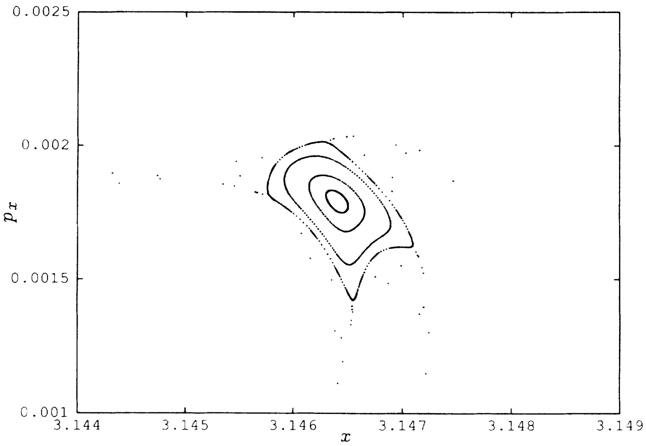

FIG. 2. Surface of section plot around the stable orbit.

face of section $y=0$ now define a two-dimensional area-preserving map $(x_{i+1},p_{x,i+1})=f(x_i,p_{x,i})$. The periodic orbits of length n are given by the fixed points of f^n, the nth iterate of f. Points of an orbit with period 11 are presented in Table I and the corresponding trajectory is shown in Fig. 1. The trace of the Jacobian over one complete cycle is $\mathrm{Tr}(J)=-0.27956\ldots$; this gives a pair of complex-conjugate eigenvalues well off the real axis and the orbit is stable. We have thus shown that the x^2y^2 potential is not purely ergodic by explicitly exhibiting a stable island—the total area of the elliptic region is about 0.005% of the surface of section area $0\le x <3.15$, $-1\le p_x\le 1$. The island is shown in Fig. 2. The equations of motion have been integrated with a fourth-order Runge-Kutta method and numerical errors are under control.

Our investigations have been much aided by studying the one-parameter family of potentials, $V(x,y)=\frac{1}{2}\times(x^2y^2)^{1/a}$, where the previous system is retained when $a=1$ and a hard four-disk scattering system is obtained in the limit $a\to 0$. Thus the periodic orbits may be labeled according to a four-disk symbolic dynamics.[16,17] The periodic orbits of the hard-disk system are transformed to the quartic system by adiabatically increasing a: $0\to 1$. Pruning of orbits due to reverse period bifurcations (period doublings, triplings, etc.) and tangent bifurcations are observed in this parameter region, and there is a temporary stabilization of periodic orbits close to the bifurcation points leading to nonhyperbolicity of the global motion. The cited stable periodic orbit is associated with a bifurcation cascade in the vicinity of $a=1$. We cannot exclude the possibility that there exist stable orbits of length shorter than 11, since the set of investigated periodic orbits of a given length is so far complete only up to period 5. Details of this study will be presented in a future publication.[17]

The apparent high periodicity of the shortest stable orbits, together with the fact they they occupy a relatively small fraction of phase space, explains why they have so long eluded detection. The key to our success was the firm control of the periodic orbit structure given by the symbolic dynamics and its pruning rules. In general, one would expect the existence of stability islands in any smooth potential with pruned symbolic dynamics.

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[1]Y. G. Sinai, Russ. Math. Surv. **25**, 137 (1970).
[2]L. A. Bunimovich and Y. G. Sinai, Math. USSR-Sb. **19**, 407 (1973).
[3]J. Hadamard, J. Math. Pures Appl. **4**, 27 (1898).
[4]E. Artin, Abh. Math. Sem. Univ. Hamburg **3**, 170 (1924).
[5]G. A. Hedlund, Bull. Am. Math. Soc. **47**, 241 (1939).
[6]M. C. Gutzwiller, Phys. Rev. Lett. **45**, 150 (1980); Phys. Scr. **T9**, 184 (1985); Contemp. Math. **53**, 215 (1986).
[7]S. G. Matanyan, G. K. Savvidy, and N. G. Ter-Arutyunyan-Savvidy, Zh. Eksp. Teor. Fiz. **80**, 830 (1981) [Sov. Phys. JETP **53**, 421 (1981)].
[8]B. V. Chirikov and D. L. Shepelyansky, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 171 (1981) [JETP Lett. **34**, 163 (1981)].
[9]E. S. Nikolaevskii and L. N. Shur, Pis'ma Zh. Eksp. Teor. Fiz. **36**, 176 (1982) [JETP Lett. **36**, 218 (1982)].
[10]G. K. Savvidy, Phys. Lett. **130B**, 303 (1983); Nucl. Phys. **B246**, 302 (1984).
[11]A. Carnegie and I. C. Percival, J. Phys. A **17**, 801 (1984).
[12]S-J. Chang, Phys. Rev. D **29**, 259 (1984).
[13]W-H. Steeb, C. M. Villet, and A. Kunick, J. Phys. A **18**, 3269 (1985).
[14]G. Sohos, T. Bountis, and H. Polymilis, Nuovo Cimento B **104**, 339 (1989).
[15]C. C. Martens, R. L. Waterland, and W. P. Reinhardt, J. Chem. Phys. **90**, 2328 (1989).
[16]B. Eckhardt and D. Wintgen, J. Phys. B **23**, 355 (1990).
[17]P. Dahlqvist and G. Russberg (to be published).