Solitary-Wave Velocity Selection in Self-Induced Transparency

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We consider self-induced transparency beyond the slowly varying envelope approximation. For a carrier wave of given frequency, we show that the possible steady-state pulse velocities form a discrete set, qualitatively changing the previous picture of self-induced transparency. The selection mechanism cannot be seen in a perturbative expansion about the slowly varying envelope approximation.

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The term self-induced transparency (SIT) was coined by McCall and Hahn in 1969. They investigated the propagation of light in nonlinear dielectrics formed by two-level systems. In such media, if the light is of low intensity, there is no transmission inside a frequency gap. However, they found that for intense enough pulses, nonlinearity can make the medium transparent again. They demonstrated[1] such SIT effects experimentally and theoretically, generating much interest and further work.[2-9]

Theoretical analyses[4-9] of SIT use a semiclassical description [Eqs. (1) and (2)] involving a number of assumptions. The dielectric (gas, semiconductor, etc.) is modeled as an ensemble of noninteracting two-level systems (atoms, excitons, etc.). The electromagnetic field is treated classically and the two-level systems semiclassically. There are no impurities or finite-temperature effects, relaxation times are infinite (no damping), nonresonant losses are absent, etc. For simplicity, one takes fields which are x and y independent; then the Maxwell wave equation becomes

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(t,z)=\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(t,z)\,. \qquad (1)$$

$\mathbf{E}$ is the electric field of magnitude $\mathcal{E}$, and $\mathbf{P}$ is the polarization due to the dipoles of density N. Each dipole moment has strength d. A true quantum dipole (an atom in a gas, an exciton in a semiconductor) has a series of energy levels. For most problems of interest, however, one can restrict oneself to the ground state (of energy zero) and first excited state (of energy $\hbar\omega_t$) for each dipole. The polarization vector has, up to a normalization constant, components u and v which satisfy the following Bloch equations[2] in the frame rotating with $\mathbf{E}$ at angular velocity $\dot{\Theta}(t)$:

$$\frac{\partial u}{\partial t}=[\dot{\Theta}(t)-\omega_t]v$$
$$\frac{\partial v}{\partial t}=-[\dot{\Theta}(t)-\omega_t]u+\kappa\mathcal{E}w,\quad \frac{\partial w}{\partial t}=-\kappa\mathcal{E}v\,, \qquad (2)$$

where $\kappa=2d/\hbar$. u and v are the in-phase (parallel to $\mathbf{E}$ or dispersive) and out-of-phase (orthogonal to $\mathbf{E}$ or absorptive) components of the macroscopic polarization $\mathbf{P}$ relative to $\mathbf{E}$, and w is the population inversion of the medium; they satisfy $u^2+v^2+w^2=1$. The atoms are predominantly in the ground state when $w\approx -1$.

McCall and Hahn considered a circularly polarized pulse with a carrier wave of frequency ω. On the time and length scale of this carrier wave, the envelope is typically slowly varying. Thus it is natural to take the slowly varying envelope approximation (SVEA),[1] where all subleading (in this limit) terms are dropped from the equations. Using this approximation, they showed that after a pulse had propagated a few classical absorption lengths into the medium, the envelope evolved into a symmetric hyperbolic-secant shape. In fact, they found a family of solitary waves with arbitrary pulse width τ and velocity $V=c/(1+2\pi\kappa\omega Nd\tau^2)$. *Solitary waves* are localized waves which are steady state, i.e., which depend on z and t only through $\zeta=t-z/V$. In 1971, Lamb[8] showed that the Maxwell-Bloch (MB) equations in the SVEA form an exactly integrable system. This meant that the propagating hyperbolic-secant pulses found by McCall and Hahn were in fact solitons. *Solitons* are solitary waves which preserve their form even if they collide.[9,10]

There have been extensions of the work of McCall and Hahn to include spatial dispersion[4(g),4(h),7(b)] and chirping.[5-7(a)] In particular, Akimoto and Ikeda[7(a)] developed a systematic perturbative expansion about the SVEA for various types of pulses based on a power-series expansion in a small parameter related to the pulse width. All these perturbative studies find pulse shapes which depend continuously on the pulse width τ so that τ is an arbitrary parameter in the problem. We shall see that this is not true of the exact steady-state pulse solutions of the MB equations: For a given carrier-wave frequency, steady-state pulses can propagate only at special parameter values, and at certain velocities, a phenomenon we call velocity selection. This has not been realized previously because it cannot be seen at any order in a perturbative expansion about the SVEA.

Let us look for steady-state solutions to the full MB equations (1) and (2). These equations have translational and rotational symmetry. Following the standard procedure for obtaining self-similar solutions to partial differential equations,[11,12] the fields are of the form

$$\mathbf{E}(t,z)=\mathcal{E}(t-z/V)\hat{\mathbf{a}}(t,z)\,, \tag{3a}$$

$$\mathbf{P}(t,z)=\tfrac{1}{2}N\hbar\kappa\{u(t-z/V)\hat{\mathbf{a}}(t,z)+v(t-z/V)\hat{\mathbf{b}}(t,z)\}\,, \tag{3b}$$

where the rotating orthogonal unit vectors are

$$\hat{\mathbf{a}}(t,z)=\begin{pmatrix}\cos\theta & 0\\ 0 & \sin\theta\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\end{pmatrix},\quad \hat{\mathbf{b}}(t,z)\perp\hat{\mathbf{a}}(t,z)\,,$$
$$\theta(t,z)=\omega t-Kz+\phi(t-z/V)\,. \tag{4}$$

This contains a phase modulation (chirping) ϕ for the electric field explicitly. The electric field is factorized into the slowly varying pulse envelope $\mathcal{E}$ and the rapidly oscillating carrier wave $\hat{\mathbf{a}}$; $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit Cartesian basis vectors. The envelope is a function of $t-z/V$, where V is the undetermined steady-state pulse velocity.

Taking components along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, Eqs. (1) and (3) give rise to two second-order coupled ordinary differential equations. As shown by Akimoto and Ikeda,[7(a)] the quantities Kc/ω and V/c satisfy conditions which can be obtained by linearizing these equations in the tail of the pulse where the excitation is very weak. It is convenient to introduce the dimensionless electric-field amplitude $E=\kappa\mathcal{E}/\omega_{\mathrm{LT}}$, where $\omega_{\mathrm{LT}}=8\pi Nd^2/\hbar$ is the gap-size frequency, and also the dimensionless time $\xi=(t-z/V)/\tau$, where τ describes the exponential rate of decay of the pulse in the tail. Then Eqs. (1)-(3) become (the overdots mean $d/d\xi$)

$$\gamma\ddot{E}-\left[\alpha+\beta\dot{\phi}+\gamma\dot{\phi}^2+\frac{s}{\Lambda}\left(\frac{s\Delta}{\Lambda}-2\right)w+\frac{s^2}{\Lambda}\dot{\phi}w\right]E=-\left(\frac{s\Delta}{\Lambda}-1\right)^2u\,, \tag{5}$$

$$\gamma\ddot{\phi}E+\left(\beta+2\gamma\dot{\phi}-\frac{s^2}{\Lambda}w\right)\dot{E}=-\left[\left(\frac{s\Delta}{\Lambda}-1\right)^2+\left(\frac{s}{\Lambda}\right)^2E^2\right]v\,, \tag{6}$$

$$\Lambda\dot{u}=(\Delta+\Lambda\dot{\phi})v\,, \tag{7}$$

$$\Lambda\dot{v}=-(\Delta+\Lambda\dot{\phi})u+Ew\,, \tag{8}$$

$$\Lambda\dot{w}=-Ev\,. \tag{9}$$

There are three independent parameters for these equations: $\Delta=(\omega-\omega_t)/\omega_{\mathrm{LT}}$, $\Lambda=1/\omega_{\mathrm{LT}}\tau$, and $s=1/\omega\tau=\Lambda/(\Delta+\omega_t/\omega_{\mathrm{LT}})$. The coefficients α, β, and γ [defined in Ref. 7(a)] as well as Kc/ω and V/c are known functions of Δ, Λ, and s. One must solve these differential equations under the boundary condition at $\xi=\pm\infty$ that the electric field vanishes, $E=0$, and the dipoles of the dielectric are in the ground state, $w=-1$.

Consider first a perturbative expansion about the SVEA. Equation (6) is a first-order linear differential equation for $\dot{\phi}$, from which one can derive a first integral:

$$\gamma\dot{\phi}E^2=-\frac{\beta}{2}E^2+\Lambda\left(\frac{s\Delta}{\Lambda}-1\right)^2(w+1)+\frac{s^2}{\Lambda}E^2w-\frac{s^2}{\Lambda}\int_0^E E'w(E')dE'\,. \tag{10}$$

This equation shows that if one knew a relation between E and w, then $\dot{\phi}(E)$ would follow. Similarly, use of Eqs. (7)-(9) would determine $u(E)$. After substitution, this would turn Eq. (5) into a nonlinear second-order differential equation for E alone. Let us take $w(E)$ to be a power series in E. Then u and $\dot{\phi}$ are also of this form, and the coefficients of these series can be determined recursively. Equation (5) then becomes an ordinary differential equation (ODE) for the electric-field amplitude E only:

$$\ddot{E}=E+C_1E^3+C_2E^5+\cdots\equiv-dV(E)/dE\,, \tag{11}$$

where the C's are calculable. To leading order (including only C_1), the solution is a hyperbolic secant, corresponding to the SVEA limit. One can do systematic expansions about this by including higher-order terms in $V(E)$. This has enabled us to derive very generally the various limits previously considered using other perturbative methods.[4-7] For instance, when $\Delta=0$, this gives the expansion in Ref. 5.

The main point of this Letter is to show that such perturbation expansions are misleading. First note that the condition $w\to-1$ and $E\to0$ at $\xi=-\infty$ specifies, except for an overall sign, the solution to Eqs. (5)-(9) everywhere modulo translations in ξ and ϕ. Thus imposing the boundary condition at both $\xi=-\infty$ and $\xi=+\infty$ overconstrains the problem, making it ill posed: In general, there are no solitary-wave solutions. In order to understand why the perturbative expansion cannot see this problem with the boundary conditions, consider the following. If one were to use Eqs. (5)-(9) to obtain an equation for E alone, it would be of sixth order. Take for illustrative purposes an equation of the form

$$\varepsilon\{E^{(6)}+\cdots\}+\ddot{E}-E+E^3=0\,. \tag{12}$$

For $\varepsilon=0$, this is of the same form as Eq. (11) for which there is a constant of motion: One has an exactly integrable system. This is indeed what happens when one takes the SVEA. An analogy for such equations is a ball rolling down a potential $V(E)$; the conserved quantity is the sum of the kinetic and potential energies. A solitary pulse corresponds to a trajectory from $E(\xi=-\infty)=0$ to $E(\xi=+\infty)=0$ which has zero total energy. When $\varepsilon=0$, energy is conserved so the trajectory leaving $E=\dot{E}=0$ is guaranteed to return to this point. However, as soon as $\varepsilon\neq0$, this is not the case, and in general, the returning trajectory misses the $E=\dot{E}=0$ point: There is no reason to expect the continuation of the solution which satisfies the boundary condition at $-\infty$ not to have any of the growing modes as $\xi \to +\infty$. Equation (12) is qualitatively different from Eq. (11) because it is of higher order: ε represents a *singular* perturbation which destroys the exact integrability. In particular, there are nonanalyticities in the solutions as $\varepsilon \to 0$. When $\varepsilon \neq 0$, one expects to have to tune the parameters Λ, Δ, and s in order to be able to satisfy the boundary conditions at both $\xi = -\infty$ and $\xi = +\infty$. This nonexistence of solitary-wave solutions cannot be seen to any order in perturbation theory in ε: At each order ($E = \sum \varepsilon^n / E_n$), the electric-field amplitude E_n satisfies a linear second-order ODE for which there is always one solution which satisfies the boundary conditions $E_n \to 0$ at $\xi = \pm\infty$. The E_n can be calculated iteratively and never signal any problem. This explains why no previous groups ever realized that expansions about the SVEA are misleading and that solitary waves generally do not exist.

To find solitary solutions, one must first make the problem well posed. We follow the procedure developed for similar boundary-value problems in other fields.[13,14] Equations (5)-(9) have translational symmetry in ξ and ϕ, and are invariant under $\xi \to -\xi$, $v \to -v$. Also, changing the sign of E, u, and v is a symmetry. Since the boundary conditions at $\xi = -\infty$ define the solution everywhere up to such translations and the sign symmetry, it is not difficult to see that solitary-wave solutions (after shifting ξ) have w and $\dot\phi$ even in ξ. Furthermore, E must be either even or odd. The standard hyperbolic-secant pulses are even, so we will restrict ourselves to this case. Then u is even and v is odd. Let us thus consider the ODEs (5)-(9) on the interval $(-\infty, 0]$ with the same boundary conditions at $\xi = -\infty$ and the condition $v = 0$ at $\xi = 0$. This new boundary problem is well posed, having in general a unique solution. Then if $\dot E = 0$ at $\xi = 0$ also, it is easy to see that one can construct a solitary-wave solution on the whole ξ axis by reflection of the solution on the half line with a change of sign for v. For $\varepsilon \neq 0$, in general $\dot E \neq 0$ at $\xi = 0$, corresponding to a solution which has some amount of growing (bad) modes as $\xi \to +\infty$. Thus the condition for existence of a solitary-wave solution is $\dot E = 0$ at the ξ where $v = 0$, and this condition can be interpreted as forbidding any cusp in E.

We numerically integrated the system Eqs. (5)-(9). For ξ in the tail, the initial conditions on the fields can be obtained from the perturbative expressions for E, $\dot E$, $\dot\phi$, u, v, and w. We evolve forward and find the ξ where $v = 0$, and then determine $\dot E$ there. Call this value $\dot E_{\rm tip}(\Delta, \Lambda, s)$. As suggested by asymptotic expansions[15] and the above arguments, in general $\dot E_{\rm tip} \neq 0$, and solitary waves do not exist for those values of Δ, Λ, s. However, we found that $\dot E_{\rm tip}$ changes sign when the parameters are varied so there are surfaces in the Δ, Λ, s space on which $\dot E_{\rm tip} = 0$. These give the parameter values for which solitary-wave solutions exist, and thus provide the selected velocities for steady-state pulses.[15,16]

We determined numerically the curves (Λ, Δ) for which there are solitary solutions at fixed $\omega_{\rm LT}/\omega_t$. As can be seen in Fig. 1, one can tune the parameters Λ, Δ, s, to obtain $\dot E_{\rm tip} = 0$ really only *inside* the gap. There are several branches of solutions which rise from $\Delta = 0$ and set at $\Delta = 1$. Other branches stop inside the gap due to the appearance of multiple solutions to $v = 0$. Full inversion ($w = 1$) at the pulse peak is never realized; rather w decreases as the pulse width increases.

For Δ outside the gap, we found that when $\Lambda \to 0$, $\dot E_{\rm tip} \approx \exp[-\lambda(\Delta, s)/\Lambda]$, where λ is a function which was found numerically. Since Λ appears in front of the derivatives in Eqs. (5)-(9), it plays the role of a singular perturbation and $\dot E_{\rm tip}$ can be thought of as the amount of bad modes as $\xi \to +\infty$. It should be zero to all orders in perturbation theory in Λ, as indeed the above form indicates. This behavior implies that there are no solitary solutions in this limit.

Our velocity selection mechanism is subtle but should be verifiable experimentally in certain systems. SIT experiments to date have not focused on obtaining steady-state pulses or on carefully measuring pulse velocities. Since steady-state pulses exist only inside the gap, one will need to be able to experimentally resolve the gap rather well; in particular any line broadening must be small compared to $\omega_{\rm LT}$. This pretty much rules out doing experiments with gases (e.g., Rb [3(b),3(c)]). However, the gap is large enough in many semiconductors to permit an experiment to test our theory. Consider, for instance, a local-optics ($m^* = \infty$) semiconductor with parameters like CdS. Take $\hbar\omega_t = 2.55$ eV and $\hbar\omega_{\rm LT} = 2.0$ meV, and assume (as in the case of CdS) that the line broadenings due to the finite relaxation times T_1 and T_2

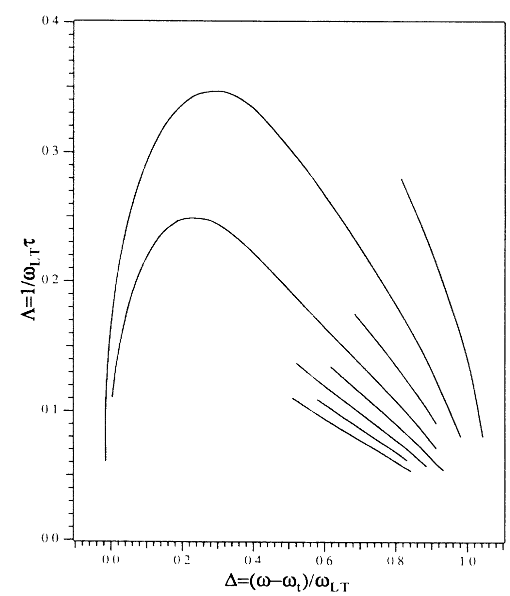

FIG. 1. Values of the parameters Λ and Δ for which solitary-wave solutions exist. We have taken $\omega_t/\omega_{\rm LT} = 1000$.

are small enough that the structure inside the gap is not washed out. Then the top branch of Fig. 1 at $\Delta=0.3$ corresponds to a pulse of width $\tau=0.95$ psec and velocity $V/c=3.4\times10^{-4}$. On the same branch at $\Delta=0.9$, we find $\tau=2.5$ psec and $V/c=3.65\times10^{-4}$, which should be measurable.

In summary, the solutions of the full MB equations differ qualitatively from the solutions within the SVEA: Steady-state pulses do not exist for arbitrary pulse width τ, but only for certain selected values which in turn determine selected velocities. Also, such solutions are solitary waves rather than solitons because the system of Maxwell-Bloch equations beyond the SVEA is not exactly integrable. Furthermore, since the pulse shape depends on Δ, contrary to the case of the SVEA, there cannot be steady-state pulses when the absorption line is inhomogeneously broadened. Comparison of our results with experiments will be of major interest. For instance, is steady-state pulse propagation feasible inside the polariton gap of a semiconductor? By tuning the laser frequency ω in the gap for a specific pulse width τ, it might be possible to observe steady-state propagation and to measure the selected velocities.

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