Self-Organized Criticality and Singular Diffusion

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We suggest that certain open driven systems self-organize to a critical point because their continuum diffusion limits have singularities in the diffusion constants at the critical point. We rigorously establish a continuum limit for a one-dimensional automaton which has this property, and show that certain exponents and the distribution of events are simply related to the order of the diffusion singularity. Numerically we show that some of these results can be generalized to include a class of sandpile models which are described by a similar, but higher-order, singularity.

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A wide variety of open dissipative dynamical systems are seen to exhibit scale-invariant characteristics reminiscent of traditional equilibrium systems at a critical point. Bak, Tang, and Wiesenfeld[1] introduced the first class of systems of this kind, the cellular automata referred to as "sandpiles." In the absence of intrinsic spatial inhomogeneities these systems generate avalanches of all sizes—the so-called "self-organized criticality" (SOC)—due to a dynamical threshold instability associated with the local slope of the pile. In an attempt to describe the anomalously large fluctuations exhibited by these and related systems, others have studied correlations in a diffusion equation with nonlinear corrections, driven by a white noise.[2] Inherent in this analysis is the *assumption* that these systems are at a critical point (associated with the critical angle of repose) and that the leading nonlinearities determine the critical exponents which describe the scaling behavior. However, a direct connection between any self-organizing model and these simple driven diffusions has not been made. So far the continuum equations which exhibit some form of scale invariance fail to capture the singular nature of the threshold criterion which governs the local evolution of the SOC models, and they do not explain the evolution of the system to a critical point.

In this Letter we explain why certain open driven systems organize to a critical point. Our key result is that continuum diffusion limits of certain self-organizing models exist and have the interesting feature of possessing diffusion "constants" which not only depend on the local density of the conserved quantity, but which in fact have a *singularity* at the critical point. With a pole in the diffusion constant, the boundary-value problem corresponding to the open driven system exhibits a boundary-layer phenomenon, and as the system size diverges the solution converges to the critical point (the pole) everywhere but on a vanishing fraction of the system. Additionally, the solution of the boundary-value problem can be used to extract information about the distribution of events in the original system. The analytical work in this paper will focus on a simple, but until now unstudied, model which exhibits SOC (nontrivial scaling properties without the tuning of a parameter). Using simulations we will show that many of our results can be generalized to describe other systems such as sandpiles.

We begin by describing the self-organizing model which succumbs to rigorous analysis. The model is designed to mimic the way that regions of high slope diffuse to regions of low slope during an avalanche.[3] We consider a finite one-dimensional lattice with a non-negative integer height $h(i)$ associated with each site $i=1,\ldots,N$. The system evolves continuously in time, such that for a fixed positive integer h_c we have the following: (i) If $h(i)\geq h_c$ then at rate 1, $h(i)\rightarrow h(i)-1$ and $h(j)\rightarrow h(j)+1$, where j is the nearest site $j>i$ with $h(j)<h_c$. (ii) The same transition rate holds for $j<i$. As in the sandpile models, the dissipation and driving mechanisms are associated with the boundary conditions of the finite system. Here we assume that the right boundary is open and the left boundary is closed, so that if there is no such j satisfying the above conditions then in (i) the grain leaves the system, and in (ii) the transition is suppressed. We drive the system by injecting new grains at rate α at site $i=1$, which instantly hop to the first site j with $h(j)<h_c$. Since configurations with $h(i)<h_c-1$ or $h(i)>h_c$ for any i are transient, each site can be in one of only two states. Taking $h_c=1$ we have a *two-state model* with site values 1 (occupied) or 0 (vacant): 1's hop at rate 1 to the nearest 0 on the right and on the left; 1's fall off the right edge, are blocked on the left edge, and are injected at the first site $i=1$ at rate α.

The advantage of working with the two-state model is that the continuum limit can be obtained rigorously for a closed system (a torus), where the spatially averaged density $\langle\rho\rangle$ of 1's is conserved. Two features make analysis of this model tractable. First, product measure with constant density $\langle\rho\rangle$ is invariant (i.e., each site is independently occupied with a 1 with probability $\langle\rho\rangle$). Second, the process is reversible (i.e., detailed balance is satisfied) for the family of equilibria ($0\leq\langle\rho\rangle\leq 1$). Note that at density $\langle\rho\rangle$, the expected jump size is (formally $N=\infty$) $\sum_{i=1}^{\infty} k\langle\rho\rangle^{k-1}(1-\langle\rho\rangle)=(1-\langle\rho\rangle)^{-1}$, which diverges as $\langle\rho\rangle\to 1$. This fact is a strong indication of a singular diffusion constant, and will play an important role in driving the open system toward the critical point. We can interpret the 0's as being domain walls which locally bound the size of a jump. The density of 0's, $p_0=1-\langle\rho\rangle$, is the order parameter in the two-state model, where $p_0\to 0$ as $\langle\rho\rangle$ approaches the critical point ($\langle\rho\rangle\to 1$). A similar, although less trivial, critical behavior was observed in sandpiles in Ref. 3.

The exact form of the diffusion constant can be deduced from a simple calculation assuming product measure and a linear profile in the density with constant slope 2ϵ. The diffusion constant is then given by $D(\rho)=I/2\epsilon$, where I is the current through a plane separating two sites:

$$I=\sum_{k=0}^{\infty}\left(\prod_{i=0}^{k}[\rho+(2i+1)\epsilon]-\prod_{i=0}^{k}[\rho-(2i+1)\epsilon]\right). \tag{1}$$

For each k the first term is associated with current flow in the positive direction, and the second term with the negative direction. The terms are the probabilities that all sites from the plane to the kth site to the left (or right) are occupied, in which case the kth particle can jump through the plane at rate 1. Keeping terms to $O(\epsilon)$ in (1) we obtain the resulting diffusion constant

$$D(\rho)=(1+\rho)/(1-\rho)^3, \tag{2}$$

which depends on the local density ρ, and has a third-order pole at $\rho=1$. With a great deal more work the following theorem can be proved, establishing that the continuum limit of the two-state model is

$$\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial x}\left(D(\rho)\frac{\partial\rho}{\partial x}\right). \tag{3}$$

Stripped of technical details, the theorem states that if we consider the system as a random measure with mass $1/N$ at each occupied site, scaling site separation by $1/N$ and the transition rates by N^2 (standard diffusion scaling), then as $N\to\infty$ this random measure approaches a deterministic density which solves the diffusion equation (3) with the diffusion constant given by (2).

Theorem.—On the discrete torus $T_N=\{1,\ldots,N\}$ denote the configuration at time N^2t by η_t, where $\eta_t(i)=1$ if site i is occupied. Given an initial density profile $\kappa(x)$ on the continuum torus $T=[0,1]$ which is uniformly bounded away from 1, start the system on T_N with this profile: $P(\eta_0(i)=1)=\kappa(i/N)$, independent of other sites. To each occupied site associate a mass $1/N$. As $N\to\infty$, with spatial rescaling this random measure converges to the deterministic density $\rho(t,x)$ which satisfies (2) and (3), with $\rho(0,x)=\kappa(x)$. The convergence is in the weak sense—for any smooth function ψ on the torus T,

$$\lim_{N\to\infty}N^{-1}\sum_{i=1}^{N}\psi\left(\frac{i}{N}\right)\eta_t(i)\to\int_T\psi(x)\rho(t,x)dx$$

in distribution.

Remark.—A complete proof of the theorem is given in Ref. 4, and is based on the techniques developed in Ref. 5 for establishing hydrodynamic limits.

Although Eq. (3) was derived for the closed system, we will use it to derive the behavior of the open driven system[6] for large N, where the appropriate boundary conditions are

$$\rho(1)=0,\quad \rho'(0)D(\rho(0))=-\alpha N, \tag{4}$$

where α is a constant. The first condition is associated with the open boundary on the right. The second condition is simply a flux-balance condition at the left boundary. Note that the driving rate is proportional to N because we rescale a system of size N into the unit interval in order to use the hydrodynamic equation (doubling the size of a one-dimensional diffusive system with fixed flux and then rescaling to the original size is equivalent to doubling the flux in the original system). Asymptotically for large N, the solution to the boundary-value problem is

$$\rho_N(x)\approx 1+\frac{1-\sqrt{4\alpha N(1-x)+1}}{2\alpha N(1-x)}. \tag{5}$$

In spite of the fact that (3) was derived for a closed system, (5) is in very good agreement with numerical results for the time-averaged configuration of the original system (see Fig. 1). The presence of the "boundary layer" at the open boundary with thickness that vanishes as $N\to\infty$ is explained by noting that for any specified density $\hat\rho<1$ and for any (large) N, there will be a site at $x_N=k_N/N$ with density approximately $\hat\rho$. The flux condition $\rho'(x_N)=-\alpha ND(\hat\rho)^{-1}$ implies that the slope of the profile at x_N diverges as N, implying the boundary-layer thickness scales like $1/N$.

Scaling properties follow immediately from the above

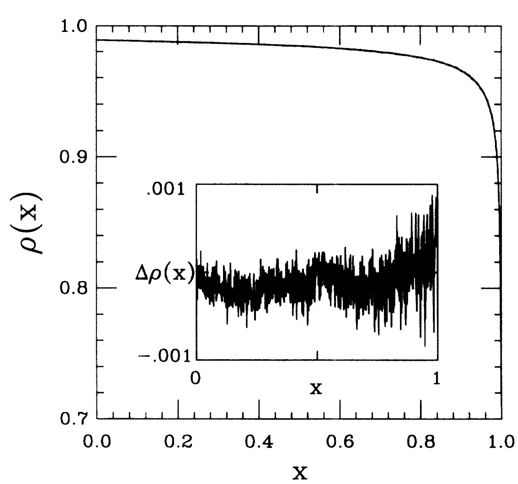

FIG. 1. The numerical and analytical profiles (both shown in the main figure) for the two-state model are essentially coincident. The inset is the difference between them. The analytical result is given by Eq. (5) with $\alpha=1$, while the numerical results are obtained as the time average of a system of corresponding size $N=8192$.

considerations: (1) The thickness Δ of the boundary layer scales like $\Delta \sim N^{-1}$. (2) The mean density $\bar{\rho}_N = \int_0^1 \rho_N(x)dx$ converges to the critical point $\rho = 1$ as $\bar{\rho}_N \sim 1 - 2/\sqrt{aN}$. This implies the density of zeros $p_0(N) = 1 - \bar{\rho}_N$ scales like $p_0(N) \sim 2/\sqrt{aN}$. (3) The steady-state distribution of jump sizes can be calculated assuming product measure locally. In a system of N sites, the probability $P(k,N)$ that a 1 will hop $k < N$ sites in an event is given by

$$P(k,N) = [1 - p_0(N)]^{k-1} p_0(N) . \tag{6}$$

The comparison between these results and simulations of the open system are excellent.

Because $p_0(N) \sim 2/\sqrt{aN}$, the distribution of events will scale with the number of sites. From (6) we derive finite-size scaling $P(k,N) = N^{-\beta} g(k/N^{\nu})$ in the limit of large N, obtaining the universal curve

$$g(z) = Ae^{-Az} , \tag{7}$$

with $A = 2/\sqrt{a}$ and $\beta = \nu = \frac{1}{2}$.

These results can be extended to other systems. For example, an obvious generalization of the two-state model to higher dimensions, in which each 1 moves to the nearest 0 in a randomly selected direction, satisfies the d-dimensional generalization of (3). The general singular diffusion equation that we consider is of the form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot [D(\rho) \nabla \rho] . \tag{8}$$

The exponents characterizing the scaling behavior depend on the order of the pole at ρ_c in the diffusion constant and on the boundary conditions, but not on any analytic factors which are bounded away from zero and infinity. Therefore, we consider the diffusion constant

$$D(\rho) = 1/(\rho_c - \rho)^{\phi} . \tag{9}$$

We will solve (8) for the simple case of a source at the center of a d-dimensional disk, where $\rho = \rho(r)$ and the boundary conditions are

$$\rho(1) = 0, \quad \rho'(0) D(\rho(0)) = -\alpha_N . \tag{10}$$

Here α_N is the rescaled driving rate which depends on both dimension and how the system is being driven. Note also that in order to have a well posed boundary-value problem in $d \geq 2$ the flux condition at $r = 0$ must be replaced by a similar condition at a radius $\delta \gtrsim 0$, which we incorporate into α_N in the solutions below: for $d = 1$,

$$\rho(r) = \begin{cases} \rho_c - [\rho_c^{-(\phi-1)} + \alpha_N(\phi-1)(1-r)]^{-1/(\phi-1)}, & \phi \neq 1, \\ \rho_c [1 - e^{-\alpha_N(1-r)}], & \phi = 1; \end{cases} \tag{11a}$$

for $d = 2$,

$$\rho(r) = \begin{cases} \rho_c - [\rho_c^{-(\phi-1)} - \alpha_N(\phi-1)\ln(r)]^{-1/(\phi-1)}, & \phi \neq 1, \\ \rho_c [1 - r^{\alpha_N}], & \phi = 1; \end{cases} \tag{11b}$$

for $d \geq 3$,

$$\rho(r) = \begin{cases} \rho_c - \left\{ \rho_c^{-(\phi-1)} - \dfrac{\alpha_N(\phi-1)}{d-2} [1 - r^{2-d}] \right\}^{-1/(\phi-1)}, & \phi \neq 1, \\ \rho_c [1 - e^{-\alpha_N(1-r^{2-d})/(d-2)}], & \phi = 1. \end{cases} \tag{11c}$$

Scaling relations are immediate. For example, in one dimension, where $\alpha_N = aN$, the density of zeros is given by

$$p_0(N) \sim \begin{cases} N^{-1/(\phi-1)}, & \text{if } \phi \neq 1,2, \\ N^{-1} & \text{if } \phi = 1, \\ N^{-1}\ln N & \text{if } \phi = 2. \end{cases} \tag{12}$$

Substituting (12) into (6), the corresponding distribution of jump sizes $P(k,N)$ is obtained (with the product-measure assumption). In each case finite-size scaling leads to a universal curve of the form (7), where the exponents β and ν are equal, and given by the same exponents as that of the order parameter p_0 in Eq. (12), with logarithmic corrections in the case $\phi = 2$. It is worth noting from (11) that, given a source at the origin, convergence to the critical point [i.e., $\rho(r) \to \rho_c$ as $N \to \infty$] occurs only if α_N diverges as $N \to \infty$. With a constant driving rate a, scaling a system with N^d sites onto the unit disk yields $\alpha_N = aN^{2-d}$, which does not diverge for $d \geq 2$. In this case there is a finite characteristic jump distance, and finite-size scaling is not observed. To observe SOC in $d \geq 2$, we could take a driving rate at the origin diverging faster than N^{d-2} in the unrescaled system, or a constant driving rate could be applied to a larger set of sites (e.g., on part of the boundary as is the case in certain sandpile models). In the latter case, one would have to solve a different boundary-value problem, but we expect that the scaling behavior would be the same as that arising from (11) with the proper α_N.

We will now provide evidence that equations of the form (8) also underlie other self-organizing models. We consider the one-dimensional limited local sandpile models[7] in which n grains of sand fall to the right whenever the local slope $z(i)$ exceeds the threshold z_c. As shown in Ref. 3, there are domain walls [any site i with $z(i) \leq z_c - n$] which bound avalanches, much like 0's bound jumps in the two-state model. When this model is run on a torus, total slope is conserved. Furthermore, there is a critical slope $s_c < z_c$ so that if the mean slope $\langle s\rangle \equiv (1/N)\sum_{i=1}^{N} z(i) < s_c$, then domain walls persist, while if $\langle s\rangle > s_c$ they become extinct. This implies that as $\langle s\rangle \uparrow s_c$ the density of domain walls approaches zero, so that the length scale on which slope is transported via avalanches diverges, suggesting a singular diffusion constant.

We verify the singular diffusion in sandpiles on both a closed and an open set, taking $z_c = n = 2$ (for which $s_c = \frac{3}{2}$). First, we start the system on a torus with average slope $\langle s\rangle$ and observe the relaxation rate of a long-wavelength sinusoidal perturbation as a function of $\langle s\rangle$. For large systems a log-log plot of relaxation rate versus $s_c - \langle s\rangle$ has a slope of -4 (inset to Fig. 2), indicating that the diffusion constant diverges as $\langle s\rangle \to s_c$ with the order of the singularity being $\phi = 4$. Second, Fig. 2 shows that the time-averaged slope profile on the open system (note the presence of a sharp boundary layer) agrees well with the solution (11) with ρ replaced by s and with $d = 1$, $\phi = 4$, and an optimally selected constant α ($a_N = aN$). From (12), we obtain the corresponding density of domain walls which scales like $N^{-1/3}$, consistent with simulations in Ref. 3. While the agreement of the profiles in Fig. 2 is impressive, so that scaling properties of the solution should apply to the system, the boundary conditions (10) do not accurately describe the sandpile model, in which there is no flux of slope through the right boundary. Instead, the boundary layer is driven by an effective potential[8] acting at the left edge which arises from an asymmetry of the rules at that boundary (see Ref. 3) and which scales with N. A similar asymmetry (potential) yields the slight decrease in the profile at the right edge.

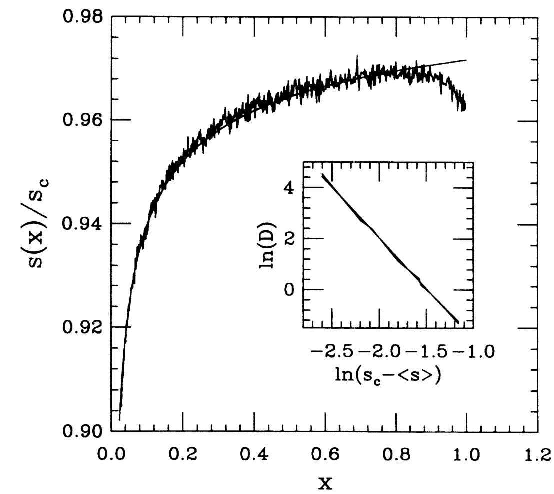

FIG. 2. Numerical and analytical slope profiles for the limited local sandpile model on an open system scaled into the unit interval ($N = 2048$). The inset illustrates the relaxation rate (i.e., the diffusion constant D since the primary mode relaxes like e^{-Dt}) of a sinusoidal perturbation as a function of distance from the critical point in the closed system. In both cases we find that the sandpile model is associated with a singular diffusion equation of the form (8), where the order of the pole $\phi = 4$ in (9).

To conclude, we have introduced a two-state model which exhibits SOC, for which we have rigorously established the existence of a diffusion limit with a singular diffusion constant. The boundary-value problem corresponding to the rescaled open driven system in one dimension has a solution which converges to the diffusion singularity as the system size diverges. In higher dimensions appropriate scaling of the driving rate yields similar results. Scaling properties follow directly from the solution. In addition, numerical evidence indicates that singular diffusion also occurs in a class of sandpile models. We suggest that singular diffusion provides a cogent framework on which to address the issue of how self-organizing systems find their critical points.

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[6]Technically the theorem does not apply to the driven system because the proof requires reversibility. Also, the theorem only holds when the density is bounded away from 1, while here we will find that the steady-state density converges to 1 like $N^{-1/2}$. Nevertheless, we expect the hydrodynamic limit to apply in the interior of the driven system since simulations show that the variance of $N^{-1}\sum_{i=1}^{N}\psi(i/N)\eta_t(i)$ vanishes as $N \to \infty$.

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[8]This suggests that with a large number of sites N, sandpiles should be approximately described by a singular diffusion equation: $\partial\rho/\partial t = \nabla\cdot[D(\rho)\nabla\rho] - \nabla[F_N(x)\rho]$, where F_N is the force due to the potential. Numerically, solutions yield profiles which are qualitatively the same as those shown in Fig. 2, but the form of F_N is unclear.