Parton Distributions from an Operator Viewpoint

Aneesh V. Manohar[(a)]

Physics Department, B-019, University of California, San Diego, La Jolla, California 92093

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Spin-dependent quark and gluon distribution functions are derived in terms of light-cone correlation functions. The first moment of the gluon asymmetry Δg is shown to be related to the Chern-Simons current K^μ. Renormalization and factorization ambiguities and their implication for the g_1 problem are discussed.

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Deep-inelastic scattering cross sections for hadron targets can be calculated in QCD using factorization. The cross section is expressed as a "hard" cross section for the scattering of a probe (such as a virtual photon or W) off a pointlike parton, convoluted with a "soft" parton distribution function which gives the probability to find the parton in the target. Schematically,[1]

$$\sigma_T^i(x,Q^2,M)=\sum_a \hat\sigma_a^i(x,Q^2,\mu)\otimes f_{a/T}(x,M,\mu)\,, \qquad (1)$$

where i is the probe, a are the various partons (quarks and gluons), M is a hadronic scale (such as $\Lambda_{\rm QCD}$ or the target mass), and μ is a renormalization scale parameter, which is usually chosen so that $\mu \gg \Lambda_{\rm QCD}$. The convolution of two functions is defined by

$$(f\otimes g)(z)\equiv\int_0^1 dx\int_0^1 dy\, f(x)g(y)\delta(z-xy)\,.$$

All the incalculable infrared effects are grouped into the parton distribution functions $f_{a/T}$. Equation (1) is nontrivial because the distribution functions $f_{a/T}$ are independent of the probe, and the hard cross sections $\hat\sigma_a^i$ are independent of the target. The hard cross sections $\hat\sigma$ are calculable in perturbation theory.

Deep-inelastic cross sections can also be computed using the operator-product expansion. This gives the structure functions in terms of certain coefficients in the operator-product expansion, multiplied by target matrix elements of towers of local twist-two quark and gluon operators. By combining the two approaches, it is possible to calculate the parton distributions in QCD in terms of light-cone correlation functions. This is well known for the quark distributions in a spin-averaged target. The results will be extended here to quark and gluon distributions in a polarized target.

Let me first summarize the standard analysis of the spin-averaged quark distribution.[2] To simplify the notation, denote the parton distribution functions in the proton, $f_{a/p}(x)$, by $a(x)$, and connected proton matrix elements by $\langle\,\rangle$. To avoid trivial complications, I will consider a single quark flavor and omit factors of the electric charge. The hadronic tensor $W_{\mu\nu}$ for virtual-photon scattering off a proton target is

$$W_{\mu\nu}(p,q)=\frac{1}{4\pi}\int d^4\xi\, e^{iq\cdot\xi}\langle[j_\mu(\xi),j_\nu(0)]\rangle\,.$$

Taking the Bjorken limit, $q^-\to\infty$, q^+ fixed, and evaluating the commutator to zeroth order in α_s (i.e., using free-field theory) gives[2]

$$F_1(x)=\frac{1}{8\pi}\int_{-\infty}^{\infty}d\xi^-\, e^{-ixM\xi^-/\sqrt2}\times\langle\bar\psi(\xi^-)\gamma^+\psi(0)-\bar\psi(0)\gamma^+\psi(\xi^-)\rangle\,.$$

The singularity as $\xi^-\to 0$ is a c-number and does not contribute to the connected matrix element. The structure function F_1 can also be computed using the parton model. To lowest order in α_s, gluonic partons do not contribute. The F_1 piece of the $\gamma^* q$ hard-scattering cross section $\hat\sigma_q^{\gamma^*}$ is $\delta(x-1)/2$ for quark or antiquark. This is equivalent to the statement that all the coefficient functions in the operator-product expansion are 1 to lowest order, since the moments of a δ function are 1. Therefore, the structure function $F_1=\hat\sigma_q\otimes(q+\bar q)$ implies that the distribution function $(q+\bar q)(x)=2F_1(x)$. Defining projection operators $P^\pm=\frac12(1\pm\alpha^3)=\frac12\gamma^\mp\times\gamma^\pm$ and $\psi^\pm=P^\pm\psi$, one gets[2]

$$(q+\bar q)(x)=\frac{1}{2\sqrt2\pi}\int_{-\infty}^{\infty}d\xi^-\, e^{-ixM\xi^-/\sqrt2}\langle(\psi^+)^\dagger(\xi^-)\psi^+(0)-(\psi^+)^\dagger(0)\psi^+(\xi^-)\rangle\,. \qquad (2)$$

The first term is the quark distribution[2]

$$q(x)=\frac{1}{2\sqrt2\pi}\int d\xi^-\, e^{-ixM\xi^-/\sqrt2}\langle(\psi^+)^\dagger(\xi^-)\psi^+(0)\rangle\,,$$

and the second term is the antiquark distribution[2]

$$\bar q(x)=\frac{1}{2\sqrt2\pi}\int d\xi^-\, e^{-ixM\xi^-/\sqrt2}\langle\psi^+(\xi^-)(\psi^+)^\dagger(0)\rangle\,.$$

It is easy to show that [2]

$$q(x),\bar q(x)\ge 0,\quad q(x),\bar q(x)=0,\quad |x|>1,$$
$$q(-x)=-\bar q(x),\quad \bar q(-x)=-q(x)\,. \qquad (3)$$

$\{q,\bar q\}$ do not vanish for $x<0$. It is convenient to work with this convention for the distribution functions, since

$\{q,\bar{q}\}$ then have definite charge-conjugation properties, and are simply related to light-cone correlation functions. $\{q,\bar{q}\}$ for $x<0$ are (up to a sign) the distribution functions for an antiproton. The even moments[3] are

$$M_n(q+\bar{q})=\frac{1}{2}\int_{-\infty}^{\infty}dx\,x^{n-1}(q+\bar{q})$$
$$=\frac{1}{2}\left(\frac{\sqrt{2}}{M}\right)^n\langle\bar{\psi}(i\partial^+)^{n-1}\gamma^+\psi\rangle \quad (4)$$

using Eq. (3). This is the same result as obtained by a direct application of the operator-product expansion. We have seen above that the moments of the hard cross section $\hat{\sigma}_q^{\gamma^*}$ give the coefficient functions in the operator-product expansion. Equation (4) implies that the moments of the parton distribution functions are the matrix elements of local operators. The parton distribution functions can also be used to find the odd moments of F_1. However, the trick of extending the integral to negative x cannot be used in this case, and the odd moments of F_1 cannot be expressed as the matrix element of local operators.

The g_1 structure function can be computed in a similar manner,

$$g_1(x)=\frac{1}{8\pi}\int d\xi^-\,e^{-ixM\xi^-/\sqrt{2}}\langle\bar{\psi}(\xi^-)\gamma^+\gamma_5\psi(0)+\bar{\psi}(0)\gamma^+\gamma_5\psi(\xi^-)\rangle\,. \quad (5)$$

The g_1 piece of the γ^*q hard-scattering cross section is $\pm\pm\delta(x-1)/2$ for a right- (left-) handed photon and a right- (left-) handed quark or antiquark. This implies that $\Delta q+\Delta\bar{q}=2g_1(x)$, where q_R ($\bar{q}_R$) and q_L ($\bar{q}_L$) are the probabilities to find right- and left-handed quarks (antiquarks) in a polarized proton, and $\Delta q=q_R-q_L$, etc. In addition to the $P^\pm$ operators introduced above, define two additional projection operators $P^{R,L}=\frac{1}{2}(1\pm\gamma_5)$. $[P^{R,L},P^\pm]=0$, so one can define fields which are simultaneous projections, $\psi^{\pm R,L}=P^\pm P^{R,L}\psi$. Using this, one can write the parton distributions

$$\Delta q(x)=\frac{1}{2\sqrt{2}\pi}\int d\xi^-\,e^{-ixM\xi^-/\sqrt{2}}\langle(\psi^{+R})^\dagger(\xi^-)\psi^{+R}(0)-(\psi^{+L})^\dagger(\xi^-)\psi^{+L}(0)\rangle\,,$$

and

$$\Delta\bar{q}(x)=\frac{1}{2\sqrt{2}\pi}\int d\xi^-\,e^{-ixM\xi^-/\sqrt{2}}\langle\psi^{+L}(\xi^-)(\psi^{+L})^\dagger(0)-\psi^{+R}(\xi^-)(\psi^{+R})^\dagger(0)\rangle\,,$$

which are the probabilities to find a net quark (antiquark) polarization in the proton. Note that L and R are interchanged between Δq and $\Delta\bar{q}$ because ψ^R annihilates left-handed quarks and creates right-handed antiquarks.

It is easy to show that the distribution functions vanish for $|x|>1$, and

$$q(x)\geq|\Delta q(x)|\geq 0,\quad \bar{q}(x)\geq|\Delta\bar{q}(x)|\geq 0,\quad \Delta q(-x)=\Delta\bar{q}(x),\quad \Delta\bar{q}(-x)=\Delta q(x)\,. \quad (6)$$

The odd moments of g_1 are given by

$$M_n(\Delta q+\Delta\bar{q})=\frac{1}{2}\int_{-\infty}^{\infty}dx\,x^{n-1}(\Delta q+\Delta\bar{q})=\frac{1}{2}\left(\frac{\sqrt{2}}{M}\right)^n\langle\bar{\psi}(i\partial^+)^{n-1}\gamma^+\gamma_5\psi\rangle\,,$$

which agrees with the operator-product expansion. The gluon distribution functions can be defined using obvious generalizations of Eqs. (2) and (5). Let us defined the gauge-invariant distributions

$$xg(x)=-\frac{1}{2\sqrt{2}M\pi}\int d\xi^-\,e^{-ixM\xi^-/\sqrt{2}}\langle G^{+\alpha}(\xi^-)G_\alpha^+(0)+G^{+\alpha}(0)G_\alpha^+(\xi^-)\rangle\,,$$

and

$$x\Delta g(x)=\frac{i}{2\sqrt{2}M\pi}\int d\xi^-\,e^{-ixM\xi^-/\sqrt{2}}\langle G^{+\alpha}(\xi^-)\tilde{G}_\alpha^+(0)-G^{+\alpha}(0)\tilde{G}_\alpha^+(\xi^-)\rangle\,, \quad (7)$$

where $G^{\mu\nu}$ is the gluon field strength, and a sum over color indices is understood. Note that

$$\lim_{x\to 0}xg(x)\neq 0,\quad \lim_{x\to 0}x\Delta g(x)=0\,,$$

so that the first moment of $\Delta g(x)$ exists, but that of $g(x)$ does not exist. Define the polarization vectors $R=(0,-1,-i,0)/\sqrt{2}$ and $L=(0,1,-i,0)/\sqrt{2}$. The sum over $\alpha=+,-,R,L$ only involves $\alpha=R,L$ since $G^{++}=G^\pm_\pm=0$;

$$-G^{+\alpha}G_\alpha^+=(G^{+R})^\dagger G^{+R}+(G^{+L})^\dagger G^{+L}\,,$$

$$iG^{+\alpha}\tilde{G}_\alpha^+=(G^{+R})^\dagger G^{+R}-(G^{+L})^\dagger G^{+L}\,,$$

using $R^\dagger=-L$, $L^\dagger=-R$. Thus $g(x)$ can be interpreted as the probability to find a gluon in the proton with momentum fraction x, and $\Delta g(x)$ as the probability to find a right-handed gluon minus the probability to find a left-handed gluon. These distributions vanish if $|x|>1$, and satisfy $g(x)\geq|\Delta g(x)|\geq 0$, $g(-x)=-g(x)$, $\Delta g(-x)=g(x)$. $M_n(g)$ for n even and $M_n(\Delta g)$ for n odd, $n>1$, are related to the matrix elements of gauge-invariant local operators,

$$M_n(g)=-\frac{1}{2}\left(\frac{\sqrt{2}}{M}\right)^n\langle G^{+\alpha}(i\partial^+)^{n-2}G_\alpha^+\rangle,$$
$$M_n(\Delta g)=\frac{i}{2}\left(\frac{\sqrt{2}}{M}\right)^n\langle G^{+\alpha}(i\partial^+)^{n-2}\tilde{G}_\alpha^+\rangle,\quad n>1, \tag{8}$$

which is also the result obtained from the operator-product expansion. Any other definition of Δg must satisfy Eq. (8), so it can differ from Eq. (7) only by $\lambda\delta(x)$ for some constant λ. There is no $\delta(x)$ singularity in the definition Eq. (7), because that would imply a ξ-independent constant in the gluon-gluon correlation function. There is no reason to expect such a term for a confining theory like QCD. Thus Eq. (7) is the preferred definition of Δg.

The moments of the quark and gluon distributions can be expressed as matrix elements of gauge-invariant, Lorentz-covariant, twist-two operators. The light-cone method naturally gives the $+\cdots+$ component of these tensors. [Nonlocal operators such as $\bar\psi(\xi)\gamma^+\psi(0)$ have to be written in general as

$$\bar\psi(\xi)\gamma^+P\exp\left[ig\int_0^\xi A_\mu(x)dx^\mu\right]\psi(0).$$

In light-cone gauge, $A^+=0$, the Taylor-series expansion of this operator gives Eq. (4).] There is no gauge-invariant local operator corresponding to the first moment of Δg in the operator-product expansion. This does not imply that the first moment of Δg is zero. We can use Eq. (7) to compute the first moment (often denoted Γ):

$$\Gamma=M_1(\Delta g)=\int_0^1 dx\,\Delta g(x)=\frac{1}{2}\int_{-\infty}^{\infty}dx\,\Delta g(x)=\frac{1}{4\sqrt{2}M}\int_{-\infty}^{\infty}d\xi^-\,\epsilon(\xi^-)\langle G^{+\alpha}(\xi^-)\tilde{G}_\alpha^+(0)-G^{+\alpha}(0)\tilde{G}_\alpha^+(\xi^-)\rangle, \tag{9}$$

where $\epsilon(z)=1$ if $z>1$, and -1 if $z<1$. This expression can be further simplified in light-cone gauge if one ignores surface terms and integrates by parts,

$$\Gamma=-\tfrac{1}{2}(\sqrt{2}/M)\langle K^+\rangle,$$

where K^μ is defined by $\partial_\mu K^\mu=G\tilde{G}$. Thus K^+ can be identified as the operator that corresponds to the first moment of Δg, provided the gauge-variant correlation function $\langle A^a(\xi^-)\tilde{G}_a^+(0)\rangle$ vanishes as $\xi^-\to\infty$ when evaluated in light-cone gauge. In the remainder of this paper, $\langle K^+\rangle$ will be used as an abbreviation for Eq. (9).

The complications of an interacting field theory have to be dealt with at first order in α_s. The moments of the structure functions can be written schematically as

$$M_n(F_1,g_1)\sim c_n^{(q)}\langle\mathcal{O}_n^{(q)}\rangle+c_n^{(g)}\langle\mathcal{O}_n^{(g)}\rangle,$$

where, as usual, the equation holds only for even moments of F_1 and odd moments of g_1. c_n are the coefficient functions, and $\mathcal{O}_n$ are twist-two quark and gluon operators. At order α_s, the quark and gluon operators mix, and need to be renormalized. The renormalization conventions used are arbitrary, but the experimentally measured structure functions F_1 and g_1 do not depend on these arbitrary conventions. Since $c_n^{(g)}$ starts at $O(\alpha_s)$, the gluon distribution is needed only at $O(1)$ and is unambiguous. The quark distribution, however, needs to be determined to $O(\alpha)$. Let $\mathcal{O}_n^{(q)}$ and $\mathcal{O}_n^{(g)}$ be the matrix elements of quark and gluon operators in a particular subtraction scheme. Then one can define the quark and gluon operators in another subtraction scheme by

$$\mathcal{O}_n'^{(q)}=\mathcal{O}_n^{(q)}+\alpha_s\lambda_n\mathcal{O}_n^{(g)},\quad \mathcal{O}_n'^{(g)}=\mathcal{O}_n^{(g)}+\alpha_s\tau_n\mathcal{O}_n^{(q)}, \tag{10}$$

where λ_n and τ_n are arbitrary constants. The coefficients in the new scheme at $O(\alpha_s)$ are given by

$$c_n'^{(q)}=c_n^{(q)},\quad c_n'^{(g)}=c_n^{(g)}-\alpha_s\lambda_n\mathcal{O}_n^{(q)}, \tag{11}$$

using $c_n^{(q)}=1$ to lowest order. We have already seen that the coefficients $c_n^{(q)}$ and $c_n^{(g)}$ are to be identified with the hard-scattering cross sections $\hat\sigma_q^{\gamma*}$ and $\hat\sigma_g^{\gamma*}$, and $\mathcal{O}_n^{(q)}$ and $\mathcal{O}_n^{(g)}$ with the distribution functions. Therefore Eqs. (10) and (11) imply that pieces of the hard-gluon cross section can be shifted into the quark distribution, and vice versa.

This ambiguity can be seen directly using factorization. As discussed in detail in Refs. 1 and 4, the total cross section to order α_s can be written as

$$\sigma_p^{\gamma*}(x,Q^2,M)=\hat\sigma_q^{\gamma*}(x,Q^2,\mu)\otimes q(x,\mu,M)+\hat\sigma_g^{\gamma*}(x,Q^2,\mu)\otimes g(x,\mu,M).$$

To identify the hard γ^*g scattering cross section, one can apply factorization to the full γ^*g scattering cross section

$$\sigma_g^{\gamma*}(x,Q^2,M)=\hat\sigma_q^{\gamma*}(x,Q^2,\mu)\otimes f_{q/g}(x,\mu,M)+\hat\sigma_g^{\gamma*}(x,Q^2,\mu)\otimes f_{g/g}(x,\mu,M). \tag{12}$$

To $O(\alpha_s)$, we can replace $f_{g/g}$ by its lowest-order value $\delta(1-x)$. Thus we can solve Eq. (12) to find

$$\hat\sigma_g^{\gamma*}(x,Q^2,\mu)=\sigma_g^{\gamma*}(x,Q^2,M)-\hat\sigma_q^{\gamma*}(x,Q^2,\mu)\otimes f_{q/g}(x,\mu,M), \tag{13}$$

and

$$\sigma_p^{\gamma*}(x,Q^2,M)=\hat\sigma_q^{\gamma*}(x,Q^2,\mu)\otimes q(x,\mu,M)+\sigma_g^{\gamma*}(x,Q^2,M)\otimes g(x,\mu,M)-\hat\sigma_q^{\gamma*}(x,Q^2,\mu)\otimes f_{q/g}(x,\mu,M)\otimes g(x,\mu,M).$$

The last term can be interpreted either as a correction to q or as a correction to $\hat\sigma_g^{\gamma*}$, since $f\otimes(g\otimes h)=(f\otimes g)\otimes h$. This is not quite correct[4] because the infrared dependence in the last term cancels the corresponding infrared dependence in $\sigma_g^{\gamma*}$ to produce the hard cross section $\hat\sigma_q^{\gamma*}$ of Eq. (13). However, any infrared-independent piece can be included in either $\hat\sigma_g^{\gamma*}$ or q; i.e., we always have the freedom to make the redefinition

$$\begin{aligned} q'(x,\mu,M)&=q(x,\mu,M)-\delta f_{q/g}(x)\otimes g(x,\mu,M),\\ \hat\sigma_g'^{\gamma*}(x,Q^2,\mu)&=\hat\sigma_g^{\gamma*}(x,Q^2,\mu)+\hat\sigma_q^{\gamma*}(x,Q^2,\mu)\otimes\delta f_{q/g}(x), \end{aligned}\tag{14}$$

which is the same ambiguity discussed earlier using local operators. There is no canonical way to fix this ambiguity. The ambiguity [to $O(\alpha_s)$] is only in the quark distribution and $\hat\sigma_q^{\gamma*}$; there is none in either the gluon distribution or the experimentally relevant quantity g_1. Any experiment that measures the gluon polarization will determine Δg as given by Eq. (7), with first moment Eq. (9).

What about $\Gamma=M_1(g_1)$? In QCD, $\Gamma=\langle\bar\psi\gamma^+\gamma_5\psi\rangle$.[5] $\bar\psi\gamma^+\gamma_5\psi$ does not mix with gluon operators, so there is no renormalization ambiguity, and one gets $\Gamma=M_1(\Delta q+\Delta\bar q)$. The parton model is not limited by any connection with local operators, so one is free to make the redefinitions Eq. (14), so that $\Gamma=M_1(\Delta q')+M_1(\delta f)\times M_1(\Delta g)$, using $M_1(\hat\sigma_q^{\gamma*})=1$ to lowest order. The additional freedom arises in the parton model because "hard" (in the sense of factorization) is not equivalent to "local" (in the sense of a local gauge-invariant operator). The prescription of Refs. 6 and 7 corresponds to $M_1(\delta f)=-\alpha_s/2\pi$, and of Ref. 8 to $M_1(\delta f)=-\alpha_s/4\pi$. Both results for $M_1(\delta f)$ are equally "correct"; there is no unique and unambiguous "anomalous gluon" contribution. Any parton-model explanation of the g_1 problem that relies on a particular value for $M_1(\delta f)$ is clearly incomplete. The natural choice for the quark distribution from the QCD viewpoint is to use $\Gamma=M_1(\Delta q)$, which retains the connection of the distribution function with gauge-invariant local operators.[9] In any case, a recent lattice calculation[10] indicates that $(3\alpha_s/2\pi)M_1(\Delta g)\lesssim 0.05$, and cannot play a significant role in the proton spin problem.

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[a]On leave from the Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139.

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