Conformal Invariance at a Deconfinement Phase Transition in (2+1) Dimensions

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The conformal dimension of the Polyakov line at the deconfinement phase transition of (2+1)-dimensional SU(2) lattice gauge theory is determined numerically using two-dimensional finite-size conformal field theory. Excellent agreement with two-dimensional Ising-model values is found for both the renormalized coupling on a spatially toroidal geometry and the conformal dimensions on a finite-width strip geometry.

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Ideas of scale invariance and conformal symmetry have proved to be strikingly powerful tools in the classification of two-dimensional phase transitions with infinite correlation length (see, e.g., Ref. 1 for a compilation of recent papers, and other references therein). Basically only one number, the conformal anomaly or central charge c, is needed to characterize a given universality class.[2,3] Scaling dimensions of all possible operators, and hence in statistical-mechanics terminology all critical exponents, then follow.

These extraordinary developments in the study of two-dimensional systems of infinite volume have been extended by Cardy and others[4,5] to more realistic systems of finite spatial extent. By a conformal mapping from the entire plane onto the surface of, for example, a cylinder of circumference L,[4] it was found that conformal symmetry completely determines, e.g., the two-point correlator even on this finite geometry. Specifically, the conformal dimensions x_i of the operators can be determined directly, and the correlation length along the infinite direction is also given explicitly. This provides us with an *exact* finite-size scaling theory, provided the assumption of conformal symmetry at the critical point is fulfilled.

The knowledge acquired about two-dimensional phase transitions through the use of conformal field theory may have ramifications in many different areas of condensed-matter physics (e.g., boundary-layer phenomena, etc.). It will also, as we show in this Letter, have consequences for the study of finite-temperature gauge theories in (2+1) dimensions. To see this, consider a lattice-regularized SU(2) gauge theory of action

$$S=\beta\sum_{\text{plaq}}(\mathrm{Tr}U_P+\mathrm{Tr}U_P^\dagger) \tag{1}$$

in (2+1) dimensions. Here $\beta=4/g^2$, and U_P denote the oriented plaquette variables in each of the three planes. Finite temperature is introduced in this Euclidean field theory by imposing periodicity in the temporal direction, the periodicity being N_τ lattice units. The temperature is then $T=1/N_\tau a$, where a is the lattice spacing. We work with an isotropic lattice, and choose units such that $a=1$.

At a certain critical temperature T_c (or, on account of 3D cutoff scaling, a certain critical coupling β_c) this gauge theory undergoes a deconfining phase transition.[6] Careful numerical studies have indicated that in this particular case the phase transition is of second order.[7-9] The universality arguments of Svetitsky and Yaffe[10] then indicate [since the center of SU(2) is Z(2)] that this phase transition should belong to the Ising fixed point *one dimension lower,* i.e., in $d=2$ dimensions.

If correct, the universality argument would imply that the finite-temperature deconfinement phase transition of this SU(2) lattice gauge theory in (2+1) dimensions can be analyzed in terms of a conformal field theory in two dimensions. In particular, the exact and universal finite-size scaling result in two dimensions can then be used to extract all pertinent information about the critical behavior of this phase transition, using with advantage what is normally a limitation of numerical simulations to *systems of finite spatial extent.* In studying the finite-size scaling around this phase transition we check simultaneously and in a highly nontrivial manner both the universality arguments of Svetitsky and Yaffe *and* the recent results stemming from the study of two-dimensional conformal field theories. We find it remarkable that two such seemingly disparate phenomena can be brought together in this fashion.

To begin, we briefly recall the finite-size scaling result of Cardy.[4] Assuming conformal symmetry at the critical point, the two-point correlation function of a scalar operator $\phi(\mathbf{r})$ can be normalized such that

$$\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle=|\mathbf{r}_1-\mathbf{r}_2|^{-2x}, \tag{2}$$

where x is the conformal (scale) dimension of ϕ. Introducing complex coordinates z on the plane, and expressing $\phi(\mathbf{r})$ in terms of these coordinates instead, i.e., $\phi(\mathbf{r}_1)=\phi(z_1,\bar{z}_1)$, one makes a conformal mapping $z\rightarrow w$, $w=(L/2\pi)\ln z$ of the full z plane onto the surface of a cylindrical geometry. Writing $w=u+iv$, where u measures the distance along the infinite direction of the cylinder, and v measures the distance around the periodic direction, the result can be given in a closed form:[4]

$$\langle \phi(u_1,v_1)\phi(u_2,v_2)\rangle = \frac{(2\pi/L)^{2x}}{\{2\cosh[(2\pi/L)(u_1-u_2)]-2\cos[(2\pi/L)(v_1-v_2)]\}^x}\,. \tag{3}$$

In the particularly interesting limiting case $|u_1-u_2| \gg L/2\pi$ one finds a simple exponential falloff,

$$\langle \phi(u_1,v_1)\phi(u_2,v_2)\rangle \sim (2\pi/L)^{2x} e^{-2\pi x|u_1-u_2|/L}\,, \tag{4}$$

which explicitly gives the form of the correlation length along the cylinder

$$\xi = L/2\pi x \tag{5}$$

in terms of the radius $L/2\pi$ and the conformal dimension x.

We are interested in the correlation among Polyakov lines, closed gauge-invariant quark loops winding around the lattice in the (also periodic) temporal direction: $W(u,v)=\frac{1}{2}\,\mathrm{Tr}\prod_\tau U_0(u,v,\tau)$, where $U_0(u,v,\tau)$ denotes a timelike link at position (u,v,τ). We choose the periodicity in the temporal direction to be $N_\tau=2$. As for the spatial sizes, we want to simulate relatively small values of the width L, and we want to mimic as closely as possible an infinite length in the orthogonal direction. As a compromise between the cost of computer time and the demand for a large extent in this other direction, we have chosen lattices of sizes $60\times L\times N_\tau$, with $L=8,12,18,24,30$, $N_\tau=2$. The limitation to $L\geq 8$ was chosen in order to avoid very-short-distance lattice artifacts.[11]

In order to proceed, we must first determine the infinite-volume critical coupling $\beta_c(\infty)$ corresponding to $N_\tau=2$. To do this, we have initially performed Monte Carlo simulations of this SU(2) lattice gauge theory on square lattices (with periodic boundary conditions) of sizes $N_\sigma^2\times N_\tau$, with $N_\tau=2$ and $N_\sigma=10$, 30, 40, and 60. We have used the Metropolis algorithm for SU(2), with four hits per link and typically 30000–60000 sweeps per point in order to determine for each finite size the almost-critical behavior. The general finite-size scaling theory[12] can be used to determine $\beta_c(\infty)$. One particularly convenient method is to construct the renormalized coupling g_r defined in terms of the normalized fourth-order susceptibility,[13]

$$g_r(\beta,N_\sigma) \equiv \chi^{(4)}(\beta,N_\sigma)/N_\sigma^2[\chi^{(2)}(\beta,N_\sigma)]^2\,, \tag{6}$$

and search for the β value at which $g_r(\beta,N_\sigma)=g_r(\beta,N_\sigma')$. In the limit $N_\sigma\to\infty$ this is easily shown to converge to a unique answer. Deviations at finite volumes are due only to subleading corrections, and it still provides a quite accurate determination of $\beta_c(\infty)$.[8,14] For these $N_\tau=2$ lattices we find $\beta_c=3.397^{+0.025}_{-0.024}$ on the basis of the accumulated statistics referred to above.[15] Although this determination of $\beta_c(\infty)$ may appear to be afflicted with a fairly large error, it is, in fact, for our purposes of absolutely sufficient accuracy.

Interestingly, already the renormalized amplitude ratio (6) converges to a *universal* number, depending only on the given universality class.[16] It follows that $g_r(\beta,N_\sigma)$ *away* from the infinite-volume critical point should be a universal function Q of the form $g_r=Q(\Delta\beta N_\sigma^{1/\nu})$, where $\Delta\beta\equiv(\beta-\beta_c)/\beta_c$. We show a test of this conventional finite-size scaling in Fig. 1. Note that g_r is bounded from below by the value -2. This corresponds to completely aligned Polyakov lines, i.e., to the totally broken phase, as can also be seen from the figure. From our numerical simulations we find $g_r\equiv g_r(\beta_c)$ at the critical point: $g_r=-1.69^{+0.06}_{-0.06}$, where the errors indicate our 95% confidence level. If we extract data from a related finite-size analysis[8] of the same theory on lattices with $N_\sigma=4$, we find $g_r\simeq -1.65$, in nice agreement with our present findings. This universal amplitude ratio can itself be computed on the basis of conformal field theory.[17] A (conformal) Schwarz-Christoffel mapping is made from the full complex plane to an arbitrary rectangle, with the resulting elliptic integrals being evaluated numerically. Knowledge of the full four-point correlator is of course also needed, but this has for the case of the Ising fixed point been computed earlier.[18] For a geometry of square lattices with "partially periodic boundary conditions" (see Ref. 17 for the precise definition) the Ising universality class corresponds to $g_r=-1.67\pm0.02$. Our number quoted above is in remarkably close agreement with this result.

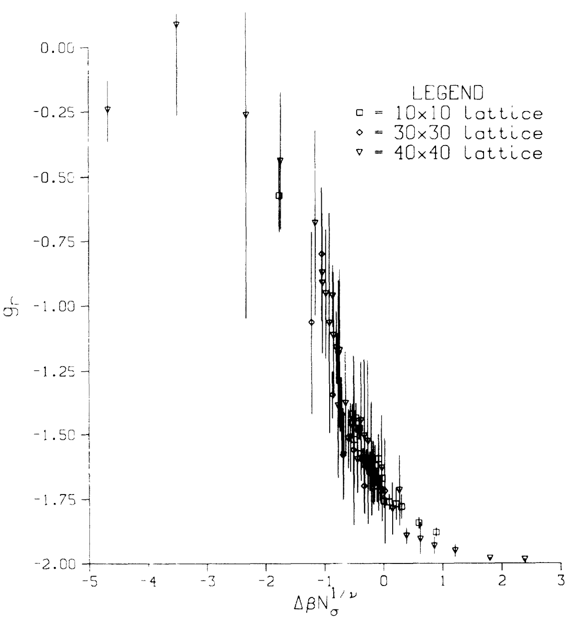

FIG. 1. The amplitude ratio (6) as a function of the scaling variable $\Delta\beta N_\sigma^{1/\nu}$ for square lattices of toroidal boundary conditions. For $N_\sigma\to\infty$ this should approach a universal scaling function.

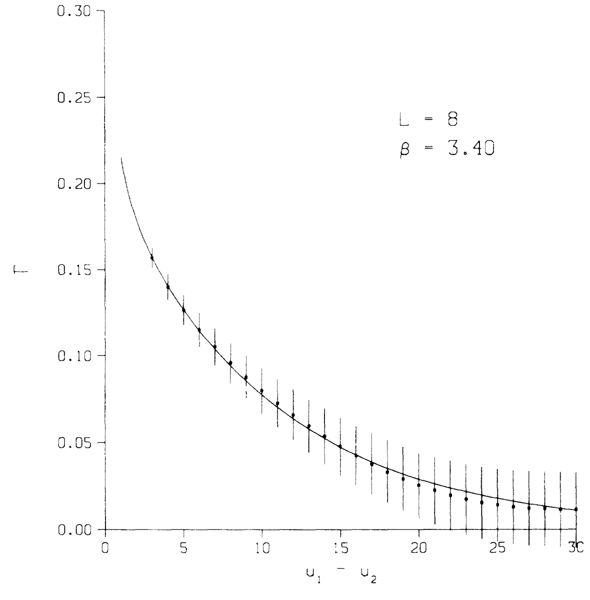

FIG. 2. The correlation function Γ for SU(2) Polyakov loops along the longitudinal direction of the cylinder, here for $L=8$ and $\beta=3.40$. The coordinates u_1-u_2 refer to the distance along the cylinder. The fully drawn curve is a least-squares fit by the form (3), giving $x=0.127^{+0.032}_{-0.026}$. The Ising value for $d=2$ dimensions is $x=\frac{1}{8}=0.125$.

Having determined $\beta_c(\infty)$ from this extensive (toroidal-geometry) finite-size analysis,[15] we now change the geometry back to that of the strip. We first keep the width L *fixed,* and measure the two-point correlation function of Polyakov loops along and across the strip. This mainly tests the expression (5) for the correlation length along the cylinder but of course fully incorporates the nontrivial structure of the exact analytic form (3). In order to see the stability of our results towards small deviations from the exact infinite-volume transition temperature, we choose to consider a sequence of couplings, $\beta=3.39$, 3.40, 3.41, and 3.42 and, for the 60×8 lattice, also $\beta=3.36$ and 3.38. A least-χ^2 fit for $L=8$ at $\beta=3.40$ gives the conformal dimension of the Polyakov loop $x=0.127^{+0.032}_{-0.026}$. In terms of the more conventionally used critical exponents, this corresponds to $\eta=2x \simeq 0.254$, or, by use of the hyperscaling relation $(2-\eta)\nu=\gamma$, $\gamma/2\nu=1-x$. The two-dimensional Ising value is $x=\frac{1}{8}=0.125$.

In Fig. 2 we show an example of the correlation measurements, here chosen solely along the length of the cylinder, and just for the case $L=8$ quoted above. The fully drawn curve corresponds to the best fit referred to above.

We can now directly observe how stable our results are for changes in β around the infinite-volume critical coupling β_c. Choosing again $L=8$, we perform corresponding fits for $\beta=3.42$, 3.41, 3.40, 3.39, 3.38, and 3.36. This yields $x=0.129^{+0.054}_{-0.041}$, $0.105^{+0.032}_{-0.026}$, $0.127^{+0.032}_{-0.026}$, $0.129^{+0.033}_{-0.027}$, $0.114^{+0.012}_{-0.011}$, and $0.173^{+0.026}_{-0.023}$, respectively. We see that apart from the trial runs at $\beta=3.36$ and 3.38, the spread in the obtained values is well within the expected statistical error, indicating that deviations at this level from not being exactly at the critical fixed-point Hamiltonian are insignificant.

A determination of the conformal dimension x for other values of L yields the numbers shown in Table I. For clarity, we have also included the results for $L=8$. Since we clearly must demand that L is much smaller than the longitudinal extent of the cylinder, we expect the deviations from the exact form (3) to grow as L increases, and, in particular, it would be surprising if even for $L=30$ we could obtain a good fit. But as can be seen from the table, in all cases the agreement with the 2D Ising value $x=\frac{1}{8}$ is quite good. The results nicely scatter around this value. The results shown in Table I correspond to all data included. Better fits with smaller errors can be obtained by making short-distance and large-distance cuts, thus reducing the effects of a finite length in the longitudinal direction and short-distance lattice artifacts.[15]

By compiling all our results for $L=8,12,18,24,30$, we can also simultaneously determine the dependence on L, and hence compare with the full form of Eq. (3). Since the L dependence can be viewed as arising independently from the numerator and the denominator of Eq. (3), we have decided to make two independent determinations of x (from the denominator) and $2x$ (from the numerator). A least-χ^2 fit combining all these data and comparing with the expected exact expression (3) gives $x=0.127^{+0.010}_{-0.009}$ (from the denominator) and $2x=0.276^{+0.024}_{-0.024}$ (from the numerator). The fact that these two determinations of x agree so closely is yet another nontrivial test of the expected behavior (3). The overall normalization is of course an unknown, since we have not

TABLE I. Numerical results for the conformal dimension x of the Polyakov line in this (2+1)-dimensional SU(2) lattice gauge theory at finite temperature.

β \ L	8	12	18	24	30
3.39	$0.129^{+0.033}_{-0.027}$	$0.192^{+0.039}_{-0.034}$	$0.157^{+0.035}_{-0.033}$	$0.192^{+0.048}_{-0.047}$	$0.110^{+0.033}_{-0.030}$
3.40	$0.127^{+0.032}_{-0.026}$	$0.125^{+0.035}_{-0.029}$	$0.082^{+0.038}_{-0.032}$	$0.159^{+0.042}_{-0.041}$	$0.177^{+0.057}_{-0.060}$
3.41	$0.105^{+0.032}_{-0.026}$	$0.107^{+0.027}_{-0.023}$	$0.172^{+0.042}_{-0.040}$	$0.143^{+0.055}_{-0.050}$	$0.154^{+0.058}_{-0.071}$
3.42	$0.129^{+0.054}_{-0.041}$	$0.126^{+0.041}_{-0.034}$	$0.128^{+0.111}_{-0.084}$	$0.102^{+0.027}_{-0.026}$	$0.133^{+0.052}_{-0.052}$

attempted (and have no need) to normalize the infinite-volume two-point function.

Independent fits can also be made to finite-size susceptibility, for which an exact result clearly can be derived once the two-point correlation function is known. We will present results for this, and other extensions of the present study, in a subsequent publication.[15]

Finally, we would like to note that these results in no way are restricted to this specific gauge group. By universality the method should apply to any $(2+1)$-dimensional finite-temperature gauge theory with a diverging correlation length at the deconfinement phase transition point. Of course, the conformal dimensions will in general be different. More details will be presented elsewhere.[15]

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