Braid of Strings

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Braiding on a string amplitude is shown to yield a Yang-Baxter relation in which the Koba-Nielsen variables play the role of spectral parameters.

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The Yang-Baxter relation plays an essential role in clarifying the interrelationship of various subjects in physics and mathematics, such as solvable statistical models, conformal field theory, knot theory, quantum groups, and soliton theory.[1-3] On the other hand, the intimate relationship between the string theory and the soliton theory has been proved.[4] Namely, the string amplitudes satisfy Hirota's bilinear difference equation,[5] a discrete analog of the two-dimensional Toda lattice. There has been, however, no argument which directly relates the string theory to the Yang-Baxter relation.

Our purpose in this paper is to study the Yang-Baxter relation associated with the braiding on the string amplitudes. The string amplitudes were formulated such that duality, the fundamental property of hadrons, is manifestly realized. Recently, many authors have discussed this property in great detail within the framework of rational conformal field theory (RCFT),[3] a very limited version of the string theory, and revealed a deep connection of this property to the quantum group and the Yang-Baxter equation. In spite of various successful results, however, this approach seems to require much further effort to explore the scope of the string model from a field-theoretical point of view. In view of this, it will be worthwhile to examine the string amplitudes themselves, which embody duality in a simple expression, and see the relationship to the Yang-Baxter equation. Therefore, in contrast to the conventional argument on RCFT, we consider the *braiding of strings* rather than their components (primary fields). This will be achieved by the use of three-string vertex operators. A simple expression for the vertex operator given in an analytic form enables us to study manifestly the duality properties of the string amplitudes[6,7] with external strings rather than external point particles.

To begin with, we first consider an integrand of the string amplitude for scattering of bosonic ground-state particles (up to some measure factor) in the operator formalism:[8]

$$F(z_1,z_2,\ldots,z_N) =\langle 0|V(k_1,z_1)V(k_2,z_2)\cdots V(k_N,z_N)g|0\rangle\,, \qquad (1)$$

where

$$V(k,z)=:e^{ihX(z)}:\equiv e^{ihX^+(z)}e^{ikX^-(z)} \qquad (2)$$

with the string coordinate $X^\mu(z)$,

$$X^{\mu+}(z)=ip^\mu \ln z+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}a_n^{\mu\dagger}z^{-n}\,,$$
$$X^{\mu-}(z)=x^\mu+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}a_n^{\mu}z^{n}\,. \qquad (3)$$

g in F denotes the vacuum operator with an arbitrary number of loops. This amplitude (1) can also be regarded as a correlation function of primary fields $V(k_j,z_j)$ whose conformal spins are given by $\frac{1}{2}k_j^2$. The Koba-Nielsen variables, the z's, are radically ordered as $|z_1|<|z_2|<\cdots<|z_N|$. In order to obtain the scattering amplitude one has to integrate (1) over z's as well as over moduli.

If $B_{i,j}$ denotes the operator which represents the monodromy of the expression (1) when z_j moves from the region $|z_i|<|z_j|$ to $|z_j|<|z_i|$ in the counterclockwise direction, it is graphically clear that[1]

$$B_{i,i+1}B_{j,j+1}=B_{j,j+1}B_{i,i+1}\ \text{ if } |i-j|\geq 2\,,\quad B_{i,j}B_{i,k}B_{j,k}=B_{j,k}B_{i,k}B_{i,j}\ \text{ if } j=i+1,\ \ k=j+1\,. \qquad (4)$$

These are the relations which characterize the braid group. If we call $F(z_1,z_2,\ldots,z_{i-1},z_i,z_{i+1},\ldots,z_N)$ the s channel, the u-channel amplitude in the operator formalism is given by

$$F_u(z_1,z_2,\ldots,z_{i-1},z_{i+1},z_i,\ldots,z_N)=\langle 0|V(k_1,z_1)\cdots V(k_{i-1},z_{i-1})V(k_{i+1},z_{i+1})V(k_i,z_i)\cdots V(k_N,z_N)g|0\rangle\,, \qquad (5)$$

so that $V(k_{i-1},z_{i-1})$ and $V(k_{i+1},z_{i+1})$ can be combined together to form resonances. So $B_{i,j}$ represents the s-u duality of the string amplitudes.

From the change of the order of V's we gain a factor

$$V(k_i,z_i)V(k_j,z_j)=R_{i,j}V(k_j,z_j)V(k_i,z_i)\,, \qquad (6)$$

where $R_{i,j}$ is given explicitly as

$$R_{i,j}=e^{i\pi\varepsilon(z_i-z_j)k_ik_j}\,, \qquad (7)$$

by using $[X(z_i),X(z_j)] = -i\pi\varepsilon(z_i - z_j)$. Here $\varepsilon(z_i - z_j)$ is the step function defined by

$$\varepsilon(z_i - z_j) = \begin{cases} +1, & |z_i| > |z_j|, \\ 0, & |z_i| = |z_j|, \\ -1, & |z_i| < |z_j|. \end{cases} \tag{8}$$

$R_{i,j}$ is clearly an explicit representation of $B_{i,j}$. Note that the dependence of $R_{i,j}$ in (7) on the variable $z_i - z_j$ is rather trivial.

Generally speaking, if the system has conformal invariance, each holomorphic piece of correlation functions (a conformal block) forms a linear space. Monodromy matrices which are independent of z's or moduli act linearly on that space. Conformal symmetry is an important symmetry which provides us with exact solvability of the system in general. But that symmetry is considered to be a stronger condition than needed for exact solvability, because the braid relation described above is a very special case of a more general Yang-Baxter relation with its dependence on spectral parameters suppressed. We are interested in the departure from conformal symmetry but keeping the exact solvability. One approach to this problem is to deform conformal field theory (CFT) so as to get a Yang-Baxter relation instead of a braid relation.

For this purpose we slightly modify the vertex operator (2) to

$$V(k,z;\rho) = e^{ikX^{+}(z;\rho)} e^{ikX^{-}(z;\rho)}, \tag{9}$$

where

$$X^{\mu +}(z;\rho) = ip^{\mu}\ln z + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n^{\mu\dagger} z^{-n} \rho^n ,$$
$$X^{\mu -}(z;\rho) = x^{\mu} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n^{\mu} z^{n} \rho^n , \tag{10}$$

with $|\rho| < 1$. Accordingly the braid operator is also modified,

$$R_{ij} = \left(\frac{z_i - \rho^2 z_j}{z_j - \rho^2 z_i} \right)^{k_i k_j} . \tag{11}$$

Let us specify (i,j,k) to be $(1,2,3)$ and write $\ln(z_3/z_1) = u$, $\ln(z_3/z_2) = v$. Then the second equation of the braid group (4) becomes the Yang-Baxter-like equation having the right dependence on u and v:

$$R_{1,2}(u-v)R_{1,3}(u)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u)R_{1,2}(u-v). \tag{12}$$

The spectral parameters are given by the Koba-Nielsen variables of the strings. Notice that these variables are associated with the spectral parameters of the inverse problem for soliton equations as is known in the soliton-string correspondence.[4] The generalization of the vertex operator given by (9) is not unique at all. This is one way of departure from the conformal symmetry, in other words, departure from the critical point of statistical models recovering the spectral parameters.[1,3] It is interesting to notice that the same expression for the modified string coordinate appears in the argument of a q-deformed string.[9] Moreover, we can show[10] that this modification leads to a quantum deformation of the Virasoro algebra, which has the structure of the Hopf algebra.

Equation (12) depends on two independent parameters. It is, however, not quite a nontrivial Yang-Baxter relation since the vector space on which it is defined is a tensor product of three one-dimensional vector spaces. A nontrivial Yang-Baxter relation is usually defined on $V \otimes V \otimes V$ with $\dim V > 1$.

In order to obtain a nontrivial relation we must go beyond the scattering amplitudes of ground-state particles. Namely, we replace the vertex operator $V(k_i,z_i)$ of (1) by the three-string vertex operator:[6]

$$W(K_i,g_i) \equiv :\exp\left[\frac{1}{2\pi} \oint \frac{dz}{z} X(g_i(x)) K_i(z) \right]: . \tag{13}$$

Here $K_i(z)$ is an operator associated to the ith external string and $g_i(z)$ is a certain analytic function regular at $z=0$. If $Y_i(z)$ is the string coordinate of the ith external string, the momentum distribution $K_i(z)$ is given by $K_i(z) = dY_i(z)/d\ln z$. This operator has been known to reproduce the Sciuto vertex[11] as we choose $g_i(z) = 1 - z$. By adding a ghost contribution to it we can obtain a physical operator which is free from spurious states.[12] We are not going to discuss details of these complications in this paper, but study the analytic structure of amplitudes given by the vertex operators of the form (13). The conformal transformations change the form of $g_i(z)$ analytically but nothing else. The duality property of the string amplitudes is manifest in the operator formalism using the vertex operator (13).

The vertex operator (13) involves all possible interactions of particles with various mass and spins in the single expression. When $K_i(z) = k_i$, the contour integral in (13) becomes $iX(g_i(0))k_i$ and W reduces to the vertex for a ground-state particle, (2), with $g_i(0) = z_i$. The interaction of a string with a particle with some particular value of spin s_i is obtained by projecting W into such a state:

$${}_i\langle s_i | W(K_i,g_i) | 0 \rangle_i , \tag{14}$$

where $|\cdot\rangle_i$ denotes a state on which K_i operates. The transition amplitude of a spin eigenstate a to another state b is calculated by further projecting (14) into such states

$$\langle b | {}_i\langle s_i | W(K_i,g_i) | 0 \rangle_i | a \rangle , \tag{15}$$

where $|\cdot\rangle$ is a state on which X operates. If the states a, b, and s_i are those of the primary fields, this vertex corresponds to the usual one of $\binom{s_i}{ba}$ type of CFT. Instead of dealing with such individual vertices with definite spins, however, we will study the simple expression W itself. In other words, we consider the string states as a whole.

In order to see the correspondence between our approach and the conventional one, we notice that if $G(g)$ denotes the operator which maps z to $g(z)$, we can write

$$W(K,g)=G(g)W(K,1)G^{-1}(g)\,. \tag{16}$$

Then the propagator between the ith and $(i+1)$th vertices is given by $G(g_i{}^{-1}g_{i+1})=G^{-1}(g_i)G(g_{i+1})$, which is a generalization of the conventional propagator $(z_{i+1}/z_i)^{L_0}$. Therefore, the states of three strings interacting at a vertex are specified by the operators K and $G(g)$, or equivalently K and g.

Let us now consider the braiding of strings. Namely, we consider the integrand (1) but all V's are replaced by W's. The ordering of the strings is specified by $g_i(0)=z_i$ radically. The braiding R_{ij} of the two strings i and j must describe the permutation of z_i and z_j. It is achieved by moving g_i and g_j to g_i' and g_j' analytically, such that the order of $g_i'(0)$ and $g_j'(0)$ is reversed (see Fig. 1). After changing the order of the corresponding W's we obtain

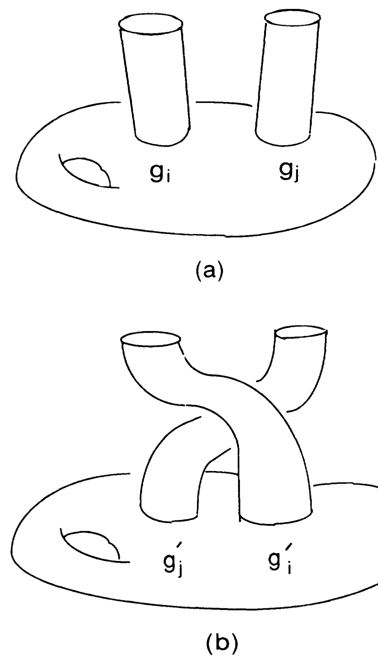

FIG. 1. (a) Unbraided strings. (b) Braided strings.

$$W(K_i,g_i)W(K_j,g_j)\to W(K_i,g_i')W(K_j,g_j')=R_{i,j}W(K_j,g_j')W(K_i,g_i')\,, \tag{17}$$

where

$$R_{i,j}=\exp\left[\left(\frac{1}{2\pi}\right)^2\oint\frac{dz}{z}\oint\frac{dw}{w}K_i(z)K_j(w)\ln\left(\frac{g_i'(z)-\rho^2 g_j'(w)}{g_j'(w)-\rho^2 g_i'(z)}\right)\right]. \tag{18}$$

Here we recall that it is sufficient to consider a Möbius map for $g_i(z)$ to study duality properties of the string amplitudes. The Möbius map is specified by three complex parameters. For instance, it can be given by

$$g_i(z)=\frac{\alpha_i z+z_i}{\beta_i z+1} \tag{19}$$

so that $g_i(0)=z_i$ is satisfied. We are now going to claim, in the rest of this paper, that this triplet of the parameters plays the role of the indices $\binom{s_i}{ba}$ specifying the vertex of RCFT. This correspondence is quite natural as we see from the structure of the vertex operator represented by (16).

Equation (17) will not give any information if g_i' and g_j' are not correlated with g_i and g_j. Suppose, on the other hand, the Möbius maps g_i and g_i' are dependent on each other, and the corresponding parameters, say β_i and β_i', are related. Then $R_{i,j}$ of (18) becomes a function of α_i,α_i' and α_j,α_j':

$$R_{i,j}=R_{i,j}\begin{pmatrix}\alpha_i & \alpha_j\\ \alpha_i' & \alpha_j'\end{pmatrix}. \tag{20}$$

β_i still remains as a free parameter. Note that we can consider this R_{ij} as a matrix specified by the suffixes (α_i,α_j) and (α_i',α_j'). Moreover, integrations over α's and β's are implied so that the amplitude remains projective invariant.

Now we combine three successive braid operations in two ways:

$$W(K_1,g_1)W(K_2,g_2)W(K_3,g_3)=R_{1,2}\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_1' & \alpha_2'\end{pmatrix}R_{1,3}\begin{pmatrix}\alpha_1' & \alpha_3\\ \alpha_1'' & \alpha_3'\end{pmatrix}R_{2,3}\begin{pmatrix}\alpha_2' & \alpha_3'\\ \alpha_2'' & \alpha_3''\end{pmatrix}W(K_3,g_3'')W(K_2,g_2'')W(K_1,g_1'')\,, \tag{21}$$

$$W(K_1,g_1)W(K_2,g_2)W(K_3,g_3)=R_{2,3}\begin{pmatrix}\alpha_2 & \alpha_3\\ \alpha_2' & \alpha_3'\end{pmatrix}R_{1,3}\begin{pmatrix}\alpha_1 & \alpha_3'\\ \alpha_1' & \alpha_3''\end{pmatrix}R_{1,2}\begin{pmatrix}\alpha_1' & \alpha_2'\\ \alpha_1'' & \alpha_2''\end{pmatrix}W(K_3,g_3'')W(K_2,g_2'')W(K_1,g_1'')\,. \tag{22}$$

Comparing (21) and (22) we obtain the relation for $R_{i,j}$:

$$R_{1,2}\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_1' & \alpha_2'\end{pmatrix}R_{1,3}\begin{pmatrix}\alpha_1' & \alpha_3\\ \alpha_1'' & \alpha_3'\end{pmatrix}R_{2,3}\begin{pmatrix}\alpha_2' & \alpha_3'\\ \alpha_2'' & \alpha_3''\end{pmatrix}=R_{2,3}\begin{pmatrix}\alpha_2 & \alpha_3\\ \alpha_2' & \alpha_3'\end{pmatrix}R_{1,3}\begin{pmatrix}\alpha_1 & \alpha_3'\\ \alpha_1' & \alpha_3''\end{pmatrix}R_{1,2}\begin{pmatrix}\alpha_1' & \alpha_2'\\ \alpha_1'' & \alpha_2''\end{pmatrix}. \tag{23}$$

In this equation $R_{i,j}$ is a function of z_i and z_j, the Koba-Nielsen variables, which specify the order of the strings. The dependence of $R_{i,j}$ on these variables is not as simple as the case of scattering of ground-state particles.

Equation (23) is the Yang-Baxter relation associated with the braiding on the string amplitudes. Before ending this paper we would like to emphasize that our result is quite general in the sense that only duality, the property which characterizes the string amplitudes, is used to derive the Yang-Baxter relation. We could derive this equation for any fraction g_i of the ith vertex as long as it is related to g_i' of the vertex in another channel. We have not even specified the relationship between g_i and g_i' which connects two different channels, but the duality is sufficient to derive the Yang-Baxter relation. The situation should be contrasted with the conventional argument of the duality in RCFT, in which a vertex operator is specified by three states of primary fields. In RCFT the duality appears as a rule under which a pair of primary fields belonging to two adjacent vertices are exchanged. It is well known that this rule is described by matrices associated with modular transformation.[3] The corresponding results will be derived in our approach by relating g_i of one channel to g_i' of another channel explicitly.

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