Correlation Functions in the One-Dimensional t-J Model

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The critical exponents for the charge, spin, electron, and superconducting correlation functions of the one-dimensional t-J model at $t=J$ are obtained for arbitrary band filling. Our method is based on the Bethe-ansatz solution and the finite-size scaling analysis in conformal field theory. The long-distance behavior of the model belongs to the same universality class as the repulsive Hubbard model. Near half filling the exponents take the values expected in the strong-correlation limit of the Hubbard model. In the low-density limit the exponents are those of the noninteracting system.

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There has been considerable interest in the t-J model since its relation with high-T_c superconductivity was pointed out.[1] In one dimension (1D), the model with $t=J$ is exactly solved by the Bethe-ansatz method.[2-4] In this paper, based on the Bethe-ansatz equations, we shall obtain the critical exponents of various correlation functions in the 1D t-J model at $t=J$ for arbitrary band filling. The exponents are calculated by using the finite-size scaling technique in conformal field theory.[5] Our results show explicitly that the model is characterized as the Luttinger liquid *à la* Haldane[6] and its fixed point is of the Tomonaga-Luttinger type.[7] The same conclusion has been drawn for the repulsive Hubbard model quite recently in both analytic[8-11] and numerical[12,13] approaches. This does not necessarily mean that both models have the same exponents. In fact, we will see that going toward the band bottom the dependences of the exponents on hole doping become considerably different from each other.

The Hamiltonian of the 1D t-J model is given by[1]

$$\mathcal{H}=-t\sum_{\langle ij\rangle,\sigma} c_{i\sigma}^{\dagger}c_{j\sigma}+J\sum_{\langle ij\rangle}(\mathbf{S}_i\cdot\mathbf{S}_j-\tfrac{1}{4}n_in_j)\,, \tag{1}$$

with an antiferromagnetic coupling $J>0$ and where $\langle ij\rangle$ denotes a nearest-neighbor pair of lattice sites. The model assumes that there is not double occupancy of every site. For the special case of $t=J$, the Hamiltonian is diagonalized by the Bethe ansatz. The Bethe-Yang transcendental equations are written in terms of the rapidities k_j and Λ_α (Refs. 2 and 3),

$$F(k_j)^{N_a}=\prod_{\beta=1}^{M}F(k_j-\Lambda_\beta),\quad j=1,\ldots,N\,,$$
$$\prod_{j=1}^{N}F(\Lambda_\alpha-k_j)=-\prod_{\beta=1}^{M}F((\Lambda_\alpha-\Lambda_\beta)/2),\quad \alpha=1,\ldots,M\,, \tag{2}$$

where $F(x)=(x+i/2)/(x-i/2)$ and M is the number of down-spin electrons among the total of N electrons on the 1D lattice with even number of sites, N_a. The rapidity distribution of the ground state consists of real k_j of unpaired electrons $(j=1,\ldots,N-2M)$ and complex $k_\alpha^{\pm}$ of spin-paired electrons $(\alpha=1,\ldots,M)$, where $k_\alpha^{\pm}$ are related to real (down-) spin rapidities Λ_α through $k_\alpha^{\pm}=\Lambda_\alpha\pm i/2$.[3] In the thermodynamic limit the rapidities k and Λ are distributed over the range $|k|>B$ and $|\Lambda|>Q$, where B and Q are determined from the electron concentration and the magnetization. Henceforth we will set $t=J=1$ for convenience.

For the excited states the "Fermi levels" of the rapidity distributions become asymmetric, say, $k<B^-$ and $k>B^+$, $\Lambda<Q^-$ and $\Lambda>Q^+$. In order to compute the finite-size corrections, however, we find it more convenient to rewrite the equations for the rapidity distributions so that the integration regions fall into the interval $B^-<k<B^+$ and $Q^-<\Lambda<Q^+$. This can be easily done by Fourier transformation. The integrations of the k and Λ distribution functions over these closed intervals yield $1-(N-M)/N_a$ and $1-N/N_a$, respectively. Accordingly, we count the excited modes in terms of upspin and hole degrees of freedom instead of down spins and electrons themselves when we deal with the excitations in this representation.

After these manipulations the finite-size contributions in the energy spectrum away from half filling are verified. We follow the method adopted in the multicomponent spin models[14] and also in the Hubbard model.[15] Here we only present our results and the details will be reported elsewhere.[16]

The correction to the ground-state energy E_0 turns out to be

$$E_0\sim\varepsilon_0N_a-\pi v_c/6N_a-\pi v_s/6N_a\,, \tag{3}$$

where ε_0 is the bulk energy density, and v_c and v_s are the velocities of the charge and spin excitations (holon and spinon), both of which are massless away from half filling. There exist several kinds of excited states which are relevant for determining the energy gaps of order $1/N_a$. For the sake of illustration let us consider the charge degrees of freedom. In the excited states the hole distribution is asymmetric, $Q^-<\Lambda<Q^+$, while in the ground state it is symmetric about the origin, $-Q<\Lambda$

$<Q$. The energy difference $N_a(Q^{\pm} \mp Q)^2$ turns out to be of order $1/N_a$, and hence gives rise to the energy gap $E-E_0 \propto 1/N_a$. We have the contribution of the change of hole number from the ground state, which is denoted by I_c (integer). The change D_c due to the particle transfer from one Fermi level of the holons to the other also participates. Notice that this process carries the momentum $(2\pi - 2k_{F\uparrow} - 2k_{F\downarrow})D_c$, where $k_{F\uparrow}$ ($k_{F\downarrow}$) is the Fermi momentum for the up- (down-) spin electrons.[4] Furthermore these excitations may be accompanied by the low-energy particle-hole excitations with small momentum transfer near the right (+) and left (−) Fermi levels, which are specified by a set of non-negative integers $N_c^{\pm}$. The spin excitations are classified in the same way, and we use the corresponding notations I_s, D_s, and $N_s^{\pm}$. When computing the gaps associated with these excitations we are naturally led to introduce the 2×2 dressed charge matrix[14,15] whose elements are given by the solutions to the integral equations ($\alpha,\beta = c,s$)

$$\xi_{\alpha\beta}(\lambda_\beta) = \delta_{\alpha\beta} + \sum_{\gamma=c,s} \int_{-q_\gamma}^{q_\gamma} \frac{d\lambda'_\gamma}{2\pi} \xi_{\alpha\gamma}(\lambda'_\gamma) K_{\gamma\beta}(\lambda'_\gamma - \lambda_\beta) , \quad (4)$$

where $\lambda_c = \Lambda$, $\lambda_s = k$ and $q_c = Q$, $q_s = B$. The kernels are defined by $K_{ss}(x) = -2(x^2+1)^{-1}$, $K_{sc}(x) = K_{cs}(x) = (x^2 + \frac{1}{4})^{-1}$, and $K_{cc}(x) \equiv 0$. The energy E and momentum P of the excited states now take the form

$$E - E_0 \sim (2\pi v_c/N_a)x_c + (2\pi v_s/N_a)x_s , \quad (5)$$

$$x_c = \left(\frac{\xi_{ss}I_c - \xi_{cs}I_s}{2\det\xi} \right)^2 + (\xi_{cc}D_c + \xi_{sc}D_s)^2 + N_c^+ + N_c^- ,$$
$$x_s = \left(\frac{\xi_{sc}I_c - \xi_{cc}I_s}{2\det\xi} \right)^2 + (\xi_{cs}D_c + \xi_{ss}D_s)^2 + N_s^+ + N_s^- , \quad (6)$$

$$P - P_0 = (2\pi - 2k_{F\uparrow} - 2k_{F\downarrow})D_c + (2\pi - 2k_{F\uparrow})D_s + \frac{2\pi}{N_a} \sum_{\alpha=c,s} (I_\alpha D_\alpha + N_\alpha^+ - N_\alpha^-) , \quad (7)$$

where the elements of the 2×2 matrix ξ are given by $\xi_{\alpha\beta}(q_\beta)$. Note that the effect of the magnetic field has been included in the results.

It is clearly seen from (3)–(7) that the charge and spin degrees of freedom are separated in the continuum limit, each of which is described by $c=1$ conformal field theory. According to the finite-size scaling theory[5] we can now read off all the critical exponents from (6), as has been done for the Hubbard model.[10] We should point out that the allowed sets of quantum numbers (I_c, I_s, D_c, D_s) are subjected to the constraints $D_c = (I_c + I_s)/2$ (mod1) and $D_s = I_c/2$ (mod1), which can be checked by taking the logarithm of (2). This condition is crucial to identify the critical exponent for a given operator. One may notice that our results look analogous to those for the Hubbard model.[10,15] We stress, however, that physical interpretation of the quantum numbers is somewhat different as we have already explained.

Let us restrict ourselves to the case of zero magnetic field. Since $B \to \infty$ for zero field the dressed charge matrix is reduced to a simple form $\xi_{cc} = \xi(Q)$, $\xi_{cs} = 0$, $\xi_{sc} = \xi(Q)/2$, and $\xi_{ss} = 1/\sqrt{2}$, where $\xi(\Lambda)$ is obtained from

$$\xi(\Lambda) = 1 + \int_{-Q}^{Q} \frac{d\Lambda'}{2\pi} G(\Lambda - \Lambda')\xi(\Lambda') , \quad (8)$$

and $G(x) = \int_{-\infty}^{\infty} d\omega \exp(-i\omega x)[1+\exp(|\omega|)]^{-1}$. It is now understood from (6) that the spin sector is described by the level-1 SU(2) Kac-Moody theory and the charge sector by free boson theory with field periodicity $\sqrt{\pi}R = \xi(Q)^{-1}$.[17] Hence the holon dressed charge $\xi(Q)$ fixes the parametrization when we bosonize the charge sector.[18] In this sense the spin sector is also mapped onto the free boson theory at periodicity $\sqrt{\pi}R = \xi_{ss} = 1/\sqrt{2}$, which corresponds to the SU(2) symmetry point on the $c=1$ critical line.

We first consider the charge-density correlation function. The asymptotic form of the equal-time correlator can be written as (neglecting logarithmic corrections)

$$\langle n(r)n(0)\rangle \sim \text{const} + A_0 r^{-2} + A_2 r^{-\alpha_s}\cos(2k_F r) + A_4 r^{-\alpha_c}\cos(4k_F r) , \quad (9)$$

where $n(r)$ is the density operator at lattice site r and $k_{F\uparrow} = k_{F\downarrow} \equiv k_F$ for zero field. Since the operator $n(r)$ changes neither the electron number nor the total spin we can set $I_c = I_s = 0$. It is seen from (6) and (7) that the $4k_F$ piece is determined by the excitation of $(D_c, D_s) = (\pm 1, 0)$, whereas the $2k_F$ piece by $(D_c, D_s) = (\pm 1, \mp 1)$. The nonoscillating part arises from the lowest particle-hole excitation. We thus find $\alpha_c = 2\xi(Q)^2$, $\alpha_s = 1 + \alpha_c/4$. One observes that both holon and spinon excitations participate in the $2k_F$ oscillation part. On the other hand, the $4k_F$ piece is dominated by the holon excitation alone, as was seen for the Hubbard model[8-10] and the Tomonaga-Luttinger model.[7] The spin-correlation function $\langle S_z(r)S_z(0)\rangle$ takes the same form as (9) except that the $4k_F$ part is absent. The critical exponent for the $2k_F$ part is equal to α_s of the density correlation. The exponent α_c is depicted in Fig. 1, where the dressed charge of the holon, $\xi(Q)$, takes the value $1 \le \xi(Q) \le \sqrt{2}$ for the electron concentration $\frac{1}{2} \ge \nu \ge 0$ ($\nu = \frac{1}{2}$ at half filling). Near half filling it behaves as $\alpha_c \sim 2 + 8(\frac{1}{2} - \nu)$. In the low-density limit we have $\alpha_c = 4$, i.e., the value for the noninteracting model. This result makes a striking contrast to the $U \to \infty$ Hubbard model, where $\alpha_c = 2$ irrespective of ν.[12,13,19]

The long-distance behavior of the electron correlation function $\langle c_\uparrow^\dagger(r)c_\uparrow(0)\rangle \sim r^{-\eta}\cos(k_F r)$ determines the exponent $\theta = \eta - 1$ of the momentum distribution function close to k_F, $\langle n_k\rangle = \langle n_{k_F}\rangle - \text{const} \times |k - k_F|^\theta \text{sgn}(k - k_F)$. Since the corresponding excitation is specified by

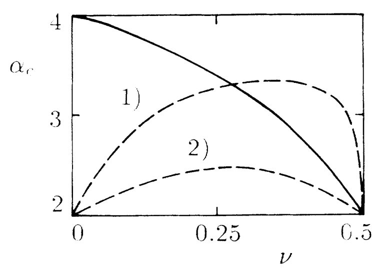

FIG. 1. The exponent α_c as a function of the electron concentration ν ($\nu = \frac{1}{2}$ for half filling). For comparison the $4k_F$ exponent in the Hubbard model is also plotted as dashed lines: (1) $U/t = 2$ and (2) $U/t = 8$ (Ref. 9). Note that $\alpha_c = 2$ for any ν in the $U \to \infty$ Hubbard model (Refs. 12, 13, and 19).

$(I_c, I_s, D_c, D_s) = (1,1,0,\pm\frac{1}{2})$, we obtain $\theta = (\alpha_c - 4)^2/16\alpha_c$. We observe that θ depends strongly on the electron concentration ν: θ decreases monotonically from $\frac{1}{8}$ to zero as ν decreases from half filling, and hence the momentum distribution varies abruptly around k_F in the low-density regime. It is interesting to compare with the $U \to \infty$ limit of the Hubbard model, where $\theta = \frac{1}{8}$ for any filling.[8-10,19]

Turning to the superconducting correlation functions we discuss the singlet and triplet pair correlations. Since double occupancy of every site is forbidden we take an intersite pair of up- and down-spin electrons for the singlet. As for the triplet an intersite pair of parallel up-spin electrons is considered. The excitations relevant for the singlet and triplet pair correlations are specified by $(I_c, I_s, D_c, D_s) = (2,1,\pm\frac{1}{2},0)$ and $(2,2,0,0)$, respectively. We then obtain the $2k_F$ oscillation piece with exponent $\beta_s = 4/\alpha_c + \alpha_c/4$ for the singlet pair. The triplet pair has the leading uniform term with exponent $\beta_t = 1 + 4/\alpha_c$. Note that the singlet pair correlation also has the uniform term with the same exponent β_t. We see that β_t decreases from 3 to 2 as ν deviates from half filling. Hence doping holes into the half-filled band enhances the superconducting correlation. The enhancement in the *t-J* model is rather conspicuous due to the large spin-exchange interaction, while it is small in the strong-correlation limit of the Hubbard model. It is also interesting to observe that even in the *t-J* model with large J $(=t)$ the spin correlation dominates the superconducting correlations for arbitrary electron filling since β_t and β_s are always larger than α_s.

These critical exponents can be expressed in terms of the bulk quantities. For the specific-heat coefficient γ the low-temperature expansion of the free energy gives $\gamma = \pi(1/v_c + 1/v_s)/3$, which corresponds to two $c = 1$ conformal theories. The compressibility and the spin susceptibility are obtained as $\chi_c = \xi(Q)^2/\pi v_c$ and $\chi_s = (g\mu_B)^2\xi_{ss}^2/\pi v_s$ with $\xi_{ss} = 1/\sqrt{2}$. We thus find $\alpha_c = 4\tilde{\chi}_c/(2\tilde{\gamma} - \tilde{\chi}_s)$, where the tilde means the corresponding quantities normalized so that $\tilde{\gamma} = \tilde{\chi}_s = \tilde{\chi}_c = 1$ in the noninteracting case. Approaching half filling χ_s remains finite (a constant of the Heisenberg model), while χ_c diverges as $(\frac{1}{2} - \nu)^{-1}$ due to the diverging density of states. Since γ is also divergent as $(\frac{1}{2} - \nu)^{-1}$, we have $\alpha_c \to 2\tilde{\chi}_c/\tilde{\gamma}$ for $\nu \to \frac{1}{2}$.

The scaling relations among α_c, α_s, θ, and β_t are those characteristic of the Tomonaga-Luttinger model.[7,13] Therefore we conclude that the *t-J* model ($t = J$), as well as the repulsive Hubbard model,[8-13] has the fixed point of Luttinger liquids.[6] Upon comparing with the large-U behavior of the Hubbard model, however, we immediately notice the considerably different ν dependence of the critical exponents in the low-density regime. For instance α_c $(=4)$ takes the value for the noninteracting system as $\nu \to 0$. This may sound a bit peculiar since the *t-J* model is supposed to be a strongly correlated system. The result implies that the hole motion in the *t-J* model is not like spinless fermions but is affected considerably by the spin fluctuation through the strong antiferromagnetic coupling J.

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