Nonlinear Resonances and Suppression of Chaos in the rf-Biased Josephson Junction

Brendan B. Plapp and Alfred W. Hübler

Center for Complex Systems Research, Department of Physics, Beckman Institute, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

(Received 9 July 1990)

The response of rf-biased Josephson junctions to special aperiodic driving forces is studied through theory and numerical simulation. It is shown that aperiodic driving forces of very small amplitude can transform the junction from a stationary state into the rotation state. In addition, it can be shown that the resulting dynamics is not chaotic, in contrast to the generic dynamics resulting from a sinusoidal driving force. We discuss possible experimental applications.

PACS numbers: 74.50.+r, 02.60.+y, 05.45.+b

The Stewart-McCumber model[1] for dynamics of rf-biased Josephson junctions is frequently used as a test case in nonlinear dynamics.[2-7] In this paper we investigate the Stewart-McCumber model with an aperiodic forcing, which models *aperiodically* biased Josephson-junction oscillators. In particular, we focus on the situation where a nonchaotic response of large amplitude emerges from a small driving force. This is called a *nonlinear resonance.*[8] The problem of finding nonlinear resonances is closely related to the problem of optimal control of nonlinear systems. Whereas extensive literature exists on the subject of linear controls with feedback and without feedback,[9] little is known about their nonlinear counterparts.[8,10,11] Recently, progress in chaos theory[8,12] had led us to a new approach to the control process.[13,14] If the experimental dynamics is given by $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{p}(t)+\mathbf{F}(t))$, where both the set of parameters $\mathbf{p}$ and the set of driving forces $\mathbf{F}$ depend only on time t, then the limiting behavior of $x(t\rightarrow\infty)$ can be made equal to a given goal dynamics $\mathbf{y}(t)$, where $\dot{\mathbf{y}}=\mathbf{g}(\mathbf{y}(t),t)$, by an appropriate F. This complete entrainment occurs if both sets of flow vectors are made equal, i.e., $\mathbf{f}(\mathbf{y},\mathbf{p}(t)+\mathbf{F}(t))=\mathbf{g}(\mathbf{y}(t),t)$ and if the special solution $\mathbf{x}(t)=\mathbf{y}(t)$ is stable.[15] As soon as the goal dynamics can be integrated in a closed form, one has a closed form for the response of the driven experimental system too, even if the driving force is aperiodic.

Considered within the Stewart-McCumber model,[1] the system to be studied consists of an ideal Josephson element of critical current I_c shunted by a capacitance c and resistance R and driven by a current source which includes a dc component of amplitude I_0 and an rf component $I_1(t)$. In terms of dimensionless parameters, the equation of motion for the junction phase ϕ is

$$\ddot{\phi}+\beta_c^{-1/2}\dot{\phi}+\sin\phi=\rho+F(t)\,, \tag{1}$$

where $\beta_c=2eI_cR^2c/\hbar$ is the Steward-McCumber parameter, ρ is the dc bias normalized to the critical current I_c, and, for a sinusoidal force, $F(t)=\rho_1\sin(\Omega t)$, with ρ_1 also normalized to the critical current and Ω the driving frequency normalized to the plasma frequency $\omega_c=(2eI_c/\hbar c)^{1/2}$. If both the friction $\eta=\beta_c^{-1/2}$ and the driving force are zero, i.e., $\eta=0$ and $F=0$, then $E=\dot{\phi}^2/2+V(\phi)$ is a constant of motion, where $V(\phi)=-\cos\phi-\rho\phi$. E is the energy of the system in units of the Josephson coupling energy $\hbar I_c/2e$.

Since the transition from a stationary state to a rotating state due to the transfer of a certain minimum energy is experimentally accessible, it is of interest to effect the transition with driving forces of small amplitude. Such an optimal driving force is determined using a variation method which seeks to minimize the mean-square amplitude of the applied rf component of the current $\bar{F}^2=(1/T)\int_0^T F^2dt$ while maintaining a fixed energy transfer to the junction given by $\Delta E=\int_0^T F\dot{\phi}\,dt$. The optimal driving force $F(t)=(\eta+\eta')\dot{\psi}$, where $\ddot{\psi}-\eta'\dot{\psi}+\sin\psi=\rho$, is obtained by a variation of F.[14] ψ is the goal dynamics in terms of control theory. $\phi(t)=\psi(t)$ is a special solution of Eq. (1). The constant η' is a Lagrange parameter.

A second variational method was also investigated. Rather than minimizing the force, the second method involves a 100% absorption of energy, i.e., the minimization of energy reflected by the Josephson oscillator. Once again, this is considered for a fixed energy input ΔE. The reflected energy is defined as

$$\Delta E_r=\int_0^T(-F\dot{\phi})\Theta(-F\dot{\phi})dt$$

where $\Theta(\cdot)$ is the step function. Since $F\dot{\phi}>0$ indicates energy that is absorbed by the junction, the choice of sign for the integrand of ΔE_r mandates that $\Delta E_r\geq 0$. However, if we assume F to be of the form $F=[\eta+\eta'(t)]\dot{\psi}$, where $\ddot{\psi}-\eta'(t)\dot{\psi}+\sin\psi=0$ is the corresponding goal dynamics, then, for the special solution of $\psi(t)=\phi(t)$, $F\dot{\phi}\geq 0$, given $\eta+\eta'(t)\geq 0$. Therefore, $\Delta E_r=0$, and we have the reflected energy at a minimum.

The only difference between the variation methods is that for the second method η' may be time dependent. This free time dependence can be used to match the requirements of an experimental apparatus. Within this context we will investigate three typical situations: (i) $\eta'=$const, (ii) the frequency changes linearly in time, and (iii) the amplitude of F is constant in time.

Figure 1 shows a typical simulation process for $\rho=0.1$, and $\eta'=\eta=1/\beta_c^{1/2}=0.01$. All of the following discussions will be made for $\eta\in[0.01,0.2]$ which are reasonable for an experiment with a real Josephson junction. For $\eta=0.2$, the oscillation is heavily damped, and for $\eta=2$, the unperturbed junction is overdamped. For $\eta=0.01$, the dynamics is already nearly conservative since the junction loses only $\sim 1\%$ of its energy per oscillation. Particularly for small η, optimal driving forces seem to be much more efficient than sinusoidal driving forces. For example, in order to switch the junction from a stationary state to a rotation, the maximum amplitude of the driving force $F_{\max}=\max F(t)$ has to exceed a critical value which depends on η. It turns out that

$$F_c=\eta[4\cos(\psi_{\min})+2\rho(2\psi_{\min}-\pi)]^{1/2} \tag{2}$$

is a good estimate for this if optimal driving forces are used, where $\psi_{\min}=\arcsin\rho$ gives the shift of the minima of V due to ρ. Figure 2 shows a comparison between F_c and $F_c^{\sin}$, the latter being the corresponding critical value for a sinusoidal driving force. Although $F_c^{\sin}$ can, in principle, be determined by algebraic methods,[5] we estimated it numerically by a systematic search through the ρ_1-Ω plane. Figure 2 shows that at $\eta=0.01$, F_c is already about 1 order of magnitude smaller than $F_c^{\sin}$, whereas for $\eta=0.2$, the difference between F_c and $F_c^{\sin}$ is less; the nonlinearity is masked by the friction.

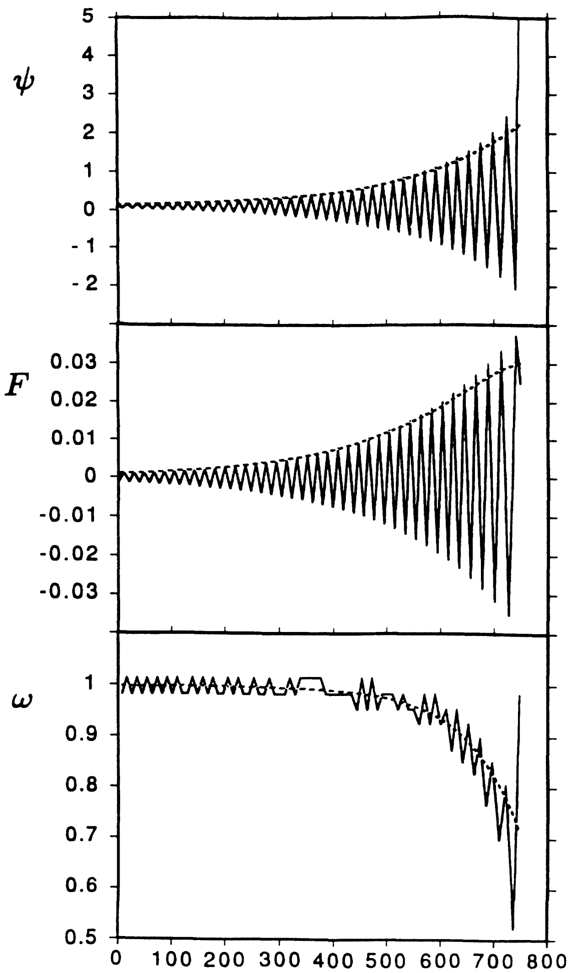

FIG. 1. The goal dynamics ψ, the optimal driving force, and its frequency vs time (——) for a fixed η'. Depicted is every third extrema. They are connected with lines. We calculate first the goal dynamics, use the resulting time series for the calculation of F, and finally integrate Eq. (1). $\psi(t)$ provides also an illustration of $\phi(t)$, since after a short transient at $t=0$, ψ is equal to ϕ within the numerical accuracy. The numerical estimate of ω results from the period between succeeding extrema of $\phi(t)$. The analytic estimates (---) for the amplitudes of the three quantities.

In our numerical investigations, we varied ρ in the range $0.1<\rho<0.9$. For large ρ the asymmetry of the oscillations increases, and higher-order Fourier amplitudes tend to increase. We will mention approximations where the magnitude of higher-order Fourier amplitudes enters. We found no qualitative changes in the response as long as ρ is not close to unity or larger than unity.[4] The magnitude of η' is chosen just large enough to transfer within ten or more oscillations enough energy to the junction to switch from a stationary state to a rotation state. The minimum amount of energy required for this operation on a conservative system is $\Delta E_{\min}=4\times\cos(\psi_{\min})+2\rho(2\psi_{\min}-\pi)$, but it can be much larger for damped systems.

Using the method of multiple scales, by eliminating all secular terms up to third order for small η' and small ψ, we obtain the following estimate for the goal dynamics (see Fig. 1):

$$\psi=\hat{\psi}(t)\sin\Phi(t)+\psi_{\min}\,, \tag{3}$$

where the amplitude of the oscillation increases exponentially, $\hat{\psi}(t)=(\psi_0-\psi_{\min})\exp(\eta' t/2)$, and where $\Phi(t):=\omega_0 t-C/\eta'\hat{\psi}^2$. $\psi_0:=\psi(t=0)$ is assumed to be close to $\psi_{\min}$. $\omega_0=[\cos(\psi_{\min})]^{1/2}$ is the eigenfrequency for small

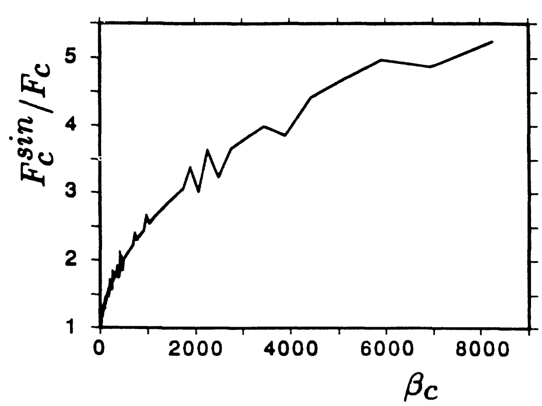

FIG. 2. The ratio of the critical amplitudes of the driving forces $F_c^{\sin}/F_c$ vs the Steward-McCumber number β_c for $\rho=0.1$.

amplitudes, and $C=(1.5+\rho^2)/24(1-\rho^2)$ represents the amplitude-frequency coupling. The frequency of ψ is given by $\omega=\dot{\Phi}=\omega_0-C\hat{\psi}^2$.

Equation (3) indicates that for a small η', $\psi(t)$ can be approximated by a sine function, for which both the amplitude and the frequency vary slowly in time compared to the typical time scale of the oscillation. If we assume that the amplitude increases exponentially up to the order of π, then we get an estimate for the stimulation time T_s, which is the minimum time required to switch the junction from a stationary state to a rotating state by an optimal driving force; $T_s \approx 2\ln[(\pi-2\psi_{\min})/\hat{\psi}(t=0)]/\eta'$. In the parameter region we investigated this estimate deviates less than 10% from the numerical value. At $\hat{\psi}=\pi-2\psi_{\min}$, V has a local maximum. As soon as $\hat{\psi}$ becomes larger the dynamics becomes a rotation.

Since the driving force F is directly proportional to ψ, the amplitude and frequency of F also vary exponentially in time. Since such a time dependence can hardly be approximated by a linear relation, it does not seem to be possible to do the experiment with present technology.

However, using the second variational method, one can find an optimal driving force where the frequency varies linearly in time. Since the basic time scale of ψ changes slowly, we can estimate the frequency shift by $\omega' \approx (d\omega/d\hat{\psi})\dot{E}/(dE/d\hat{\psi})$. $\dot{E}$ can be estimated by $\dot{E} \approx 2\eta' E$ if we assume that higher-order Fourier amplitudes of ψ fall off rapidly. Using these approximations, η' reads

$$\eta'(t)=-\omega'\frac{12\{\sin[\hat{\psi}(t)+\psi_{\min}]-\rho\}\omega_0^3}{(1.5+\rho^2)\hat{\psi}(t)E(t)} \tag{4}$$

for a fixed frequency shift $\omega'<0$.

Figure 3 shows such a stimulation. The numerical simulations indicate that for this type of excitation the amplitude of the driving force approaches a constant value close to F_c [Eq. (2)] in the limit of large t. This property makes them favorable for experiments with present technology since the frequency shift is, by definition, kept fixed, and the time dependence might be approximated by a constant equal to the limiting value of the exact driving force. The amplitude of the driving force can be estimated by

$$\hat{F}(\psi,\omega',\eta) \approx [\eta+\eta'(\hat{\psi},\omega')][2E(\hat{\psi})]^{1/2}.$$

It has a maximum during the first few oscillations, reaches a minimum, and then approaches a value close to F_c [Eq. (2)]. For large ω', this minimum vanishes and $\hat{F}$ simply decreases in time. For small ω', the amplitude during the first few oscillations is smaller than F_c, but at a certain ω_c', it exceeds this value. If the amplitude during the first few oscillations, for example, is estimated by $\hat{F}_f(\omega',\eta)=\hat{F}(\pi/6,\omega',\eta)$, we get from $\hat{F}_f(\omega_c',\eta)=F_c$ a good estimate for ω_c'. If we use $\dot{\omega}=-C\,d\hat{\psi}^2/dt=\omega'$, we obtain from $\hat{\psi}(T_s)=\pi-2\psi_{\min}$ a good estimate for the stimulation time,

$$T_s \approx -[(\pi-2\psi_{\min})^2-(\psi_0-\psi_{\min})^2]C/\omega'. \tag{5}$$

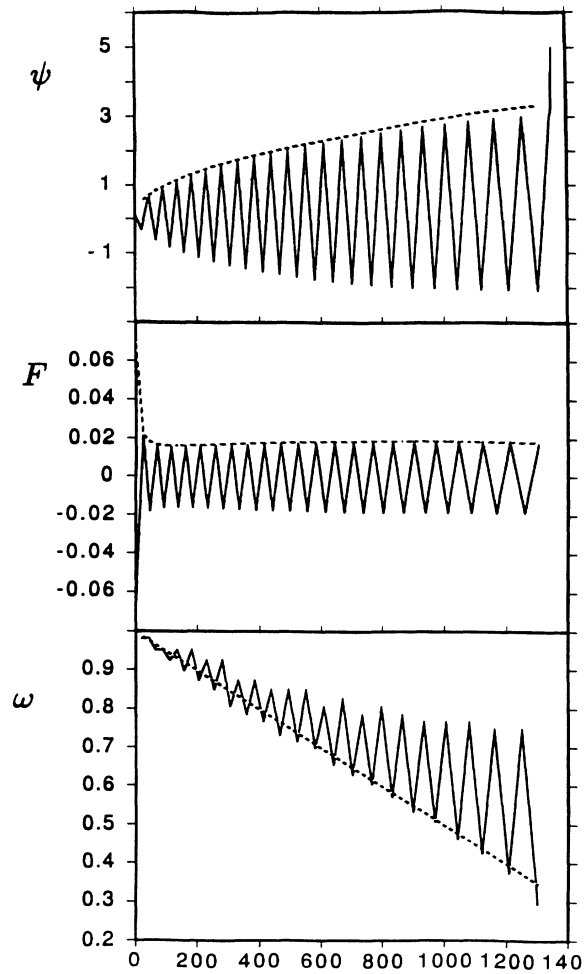

FIG. 3. The goal dynamics ψ, the optimal driving force, and its frequency vs time (——) for a fixed frequency shift $\omega'=0.0005$, where $\beta_c=10000$, and $\rho=0.1$. Depicted is every seventh extrema. They are connected with lines. Because of the asymmetry of the oscillation the numerical value for the frequency oscillates. The analytic estimates (---) for the amplitudes of the three quantities.

Certainly it is also possible to calculate driving forces which satisfy the second variation method and where the amplitude of F is kept at a constant level $F_{\max}$. This can be achieved by $\eta'(t)=(F_{\max}-F_c)(2E)^{-1/2}$. $F_{\max}$ has to exceed F_c [Eq. (2)] in order to get a growing amplitude. In this case we find that for a certain range of $F_{\max}$, the frequency decreases approximately linearly in time (for example, $\eta=0.01$, $\rho=0.1$, and $F_{\max}=0.019$), whereas for larger or smaller values of $F_{\max}$, the frequency-time relation describes a curve. The driving forces with a constant amplitude and a linear frequency shift $F(t)=F_{\max}\times\sin[(\omega_0-\omega' t/2)t]$ represent a compromise between

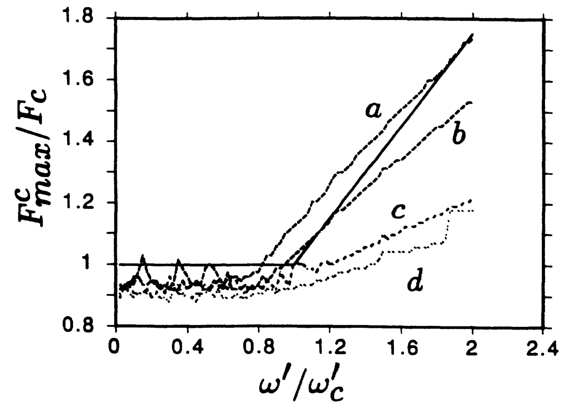

FIG. 4. The critical amplitude $F^c_{\max}$ vs the frequency shift ω' in dimensionless units for $\beta_c = 10000$, 2500, 100, and 25, curves a–d, and $\rho = 0.1$. The theoretical estimate (——).

both constraints, the constant amplitude and the constant frequency shift.

Therefore, we investigated those driving forces in detail. They are applied for a time of length T_s. For T_s we used the estimate given by Eq. (5). Extended numerical investigations show that $F_{\max}$ has to exceed both F_c and $\hat{F}_f$. As long as ω' is smaller than ω'_c, F_c is larger than $\hat{F}$. Therefore, the minimum amplitude $F^c_{\max}$ necessary to transform the system from a stationary state to a rotation state is independent of ω' and given by F_c (see Fig. 4). Above ω'_c, the limit increases at a rate that can be roughly estimated by $F^c_{\max} = \hat{F}(\pi/6, \omega', \eta)$. Once $\hat{F}$ is larger than these limits, the amplitude and the frequency shift of the junction are very similar to the response to an optimal driving froce with fixed frequency shift. Especially for small ω', which is the experimentally interesting case, a driving force with a fixed frequency shift and an amplitude equal to or larger than F_c [Eq. (2)] seems to stimulate the junction.

This work has been supported by the Aspen Center for Physics, Aspen, Colorado, and the Institute for Scientific Interchange, Torino, Italy. We would like to thank D. J. VanHarlingen for discussions.

[1] W. C. Stewart, Appl. Phys. Lett. **12**, 277 (1968); D. E. McCumber, J. Appl. Phys. **39**, 3113 (1968).

[2] For a review on algebraic methods, see M. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion*, Springer Series in Applied Mathematical Sciences Vol. 38 (Springer-Verlag, New York, 1983).

[3] B. A. Huberman, J. P. Crutchfield, and N. Packard, Appl. Phys. Lett. **37**, 750 (1980).

[4] D. D'Humieres, M. R. Beasley, B. A. Huberman, and A. Libchaber, Phys. Rev. A **26**, 3483 (1982).

[5] R. L. Kautz and J. C. Macfarlane, Phys. Rev. A **33**, 498 (1986), and references therein.

[6] W.-H. Steeb and A. Kunick, Phys. Rev. A **25**, 2889 (1982).

[7] T. Geisel, J. Nierwetberg, and A. Zacherl, Phys. Rev. Lett. **54**, 616 (1985).

[8] G. Reiser, A. Hübler, and E. Lüscher, Z. Naturforsch. **42a**, 803 (1987).

[9] See, for example, B. D. O. Anderson, *Stability Analysis of Adoptive Systems: Passivity and Averaging Analysis* (MIT, Cambridge, MA, 1986).

[10] A. P. Peirce, M. A. Dahleh, and H. Rabitz, Phys. Rev. A **37**, 4950 (1988).

[11] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[12] J. D. Farmer and J. J. Sidorowich, Phys. Rev. Lett. **59**, 845 (1987).

[13] A. Hübler and E. Lüscher, Naturwissenschaften **76**, 67 (1989).

[14] E. Lüscher and A. Hübler, Helv. Phys. Acta **62**, 544 (1989).

[15] E. A. Jackson and A. Hübler, "Periodic Entrainment of Chaotic Logistic Map Dynamic" (to be published).