Inconsistency of Scale-Invariant Curvature Coupled to Gravity

D. Zoller
Physics Department, University of Florida, Gainesville, Florida 32611
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We show that the scale-invariant curvature action for paths, the point-particle version of Polyakov's extrinsic-curvature action for surfaces, does not couple consistently to gravity. The curvature action for paths yields a massless representation of the Poincaré group with fixed helicity and so potentially provides a description of single photons and gravitons. We present a physical interpretation of the inconsistency in terms of the nonlocalizability of the photon and point out a conceptual kinship with the local supersymmetry of a spinning particle.

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Scale-invariant higher-derivative actions are invoked as necessary to study the phase structure of theories of random paths[1] and surfaces.[2] For the case of surfaces, Polyakov[3] has suggested that the scale-invariant extrinsic-curvature action is relevant to QCD, quantum gravity, and the three-dimensional Ising model. The scale-invariant action for paths,[4] aside from shedding light on the more complicated analog for surfaces, is interesting in its own right. It provides a dynamical system potentially describing a single photon or graviton as it gives rise to a massless representation of the Poincaré group[5] of fixed helicity. Since gravity is a universal force, it is necessary to consider the coupling to gravity. We shall establish that the natural coupling of the scale-invariant curvature theory to gravity does not yield a consistent theory.

The inconsistency arises from an unusual symmetry feature of the theory. The scale-invariant curvature theory exhibits a hidden local symmetry in that the number of constraints is larger than one would expect on the basis of reparametrization invariance alone. The coupling to gravity, fixed by the requirement of reparametrization invariance and general covariance, does not obviously respect the hidden local symmetry. Indeed, we shall show that the constraint algebra no longer closes in the presence of curved spacetime.

The action that defines the scale-invariant curvature theory of paths is

$$S=h\int ds\,\kappa\,, \tag{1}$$

where h is a dimensionless constant and κ is the curvature. In flat spacetime the curvature is the magnitude of the acceleration with respect to arc length

$$\kappa=\left[\left(\frac{d^2x}{ds^2}\right)^2\right]^{1/2} \tag{2}$$

and in an arbitrary parametrization is

$$\kappa=[\ddot{x}^2\dot{x}^2-(\ddot{x}\cdot\dot{x})^2]^{1/2}/(\dot{x}^2)^{1/2}\dot{x}^2\,. \tag{3}$$

In curved spacetime $g_{\mu\nu}$ the curvature is again the magnitude of the acceleration

$$\kappa=(g_{\mu\nu}D^2x^\mu D^2x^\nu)^{1/2} \tag{4}$$

but with covariant derivatives

$$D^2x^\mu=\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\,, \tag{5}$$

where $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol. The equations of motion in flat spacetime are simple in arc-length gauge and helixes about null lines are solutions. The helical motion provides a natural model of massless spinning particles.

The theory must be recast in the Hamiltonian form to discuss the representation of the Poincaré group and to easily investigate the consistency of the constrained system. Ostragradsky first formulated such higher-derivative theories in the Hamiltonian form[6] and the generalized canonical coordinates are

$$q'=\dot{q},\quad p'=\frac{\partial L}{\partial\ddot{q}},\quad p=\frac{\partial L}{\partial\dot{q}}-\frac{d}{dt}\left(\frac{\partial L}{\partial\ddot{q}}\right), \tag{6}$$

where the basic dynamical variable is q. The Hamiltonian is

$$H=pq'+p'\dot{q}'-L \tag{7}$$

and p,q and p',q' are canonically conjugate phase-space variables. The phase space is doubled as is appropriate for a fourth-order differential equation.

The Hamiltonian form of the theory in flat spacetime consists of five constraints

$$\begin{aligned}&P\cdot X'=0,\quad P'\cdot X'=0\,,\\ &P'^2X'^2-h^2=0,\quad P\cdot P'=0,\quad P^2=0\,,\end{aligned} \tag{8}$$

where P,X and P',X' are canonically conjugate pairs. The first three constraints are primary and the last two constraints are secondary and tertiary. There are no further constraints as the constraint algebra closes. Noether's theorem applied to the local symmetry of reparametrization invariance implies that the Hamiltonian vanishes and $p'q'=0$. The theory therefore exhibits a hidden local symmetry in that there are five constraints rather than the two implied by Noether's theorem. Theories based on powers of the curvature different from one have only the two constraints[7] associated with reparametrization invariance. It would be interesting to understand the three extra constraints as a local symmetry of the scale-invariant Lagrangian.

The Noether charges associated with Poincaré symmetry P^μ,

$$M^{\mu\nu}=X^{[\mu}P^{\nu]}+X'^{[\mu}P'^{\nu]}\,, \tag{9}$$

form a representation of the Poincaré Lie algebra and the values of Casimirs can be calculated from the constraints. The four-momentum squared and the Pauli-Lubanski vector squared are zero. The helicity is the coupling constant h of the curvature theory similar to the way in which mass is the coupling constant of the arc-length theory. The representation of the Poincaré group is therefore irreducible.

We potentially have a new description of photons or gravitons where both momentum and spin arise dynamically from motion. To test whether such a description is valid we must introduce interactions. One would like, for example, to compute the bending of a single photon in the gravitational field of the Sun. As it is not evident how to gauge fix the hidden local symmetry in the Euler-Lagrange equations, we will study the Hamiltonian formulation of the curvature theory in a curved background spacetime $g_{\mu\nu}$.

The Hamiltonian constraints of the arc-length theory of paths and the area theory of surfaces in a curved background can be obtained from the corresponding flat-spacetime constraints by simply replacing the flat metric $\eta_{\mu\nu}$ by the curved one $g_{\mu\nu}$. Such a replacement, however, is not correct for the curvature theory. The canonical momentum P^μ does not transform like a contravariant vector and so simply replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}$ does not give generally covariant constraints. The combination

$$P_\alpha - P'\Gamma(X')_\alpha\,, \tag{10}$$

where

$$P'\Gamma(X')_\alpha = P'_\mu \Gamma^\mu_{\alpha\beta} X'^\beta\,, \tag{11}$$

does transform like a contravariant vector. The generally covariant primary constraints are

$$\begin{gathered} P'\cdot X'=0, \quad P'^2X'^2-h^2=0\,, \\ (P-P'\Gamma(X'))\cdot X'=0\,. \end{gathered} \tag{12}$$

The secondary and tertiary constraints, respectively, are

$$\begin{gathered} (P-P'\Gamma(X'))\cdot P'=0\,, \\ (P-P'\Gamma(X'))^2 - R_{\mu\nu\alpha\beta}X'^\mu P'^\nu X'^\alpha P'^\beta=0\,, \end{gathered} \tag{13}$$

where the Reimann tensor contracted into the internal angular momentum arises from the Poisson bracket

$$R_{\mu\nu\alpha\beta}X'^\mu P'^\nu X'^\alpha P'^\beta = [(P-P'\Gamma(X'))_\mu,(P-P'\Gamma(X'))_\nu]X'^\mu P'^\nu\,. \tag{14}$$

The Poisson bracket of the tertiary constraint with the last primary constraint gives

$$R_{\mu\nu\alpha\beta;\gamma}X'^\mu P'^\nu X'^\alpha P'^\beta X'^\gamma - 4R_{\mu\nu\alpha\beta}X'^\mu(P-P'\Gamma(X'))^\nu X'^\alpha P'^\beta \tag{15}$$

and so the constraint algebra does not close in a general curved background spacetime. Although the constraint algebra does close in the special case of isotropic spacetimes where the Riemann tensor is proportional to the appropriately antisymmetrized product of two metrics, restriction to such spacetimes excludes the physically interesting case of the Schwarzschild geometry. Further, Poisson brackets lead in general to an infinite number of constraints with arbitrarily high derivatives of the Riemann tensor. Such a situation is inconsistent in the 16-dimensional phase space. Five constraints in the 16-dimensional phase space account precisely for the $6=16-2\times5$ continuous degrees of freedom of a photon. The mere fact that there are more than five constraints in curved spacetime, much less an infinite number, implies we do not have a viable description of the photon.

The inconsistency displayed here agrees with the well-known difficulty of massless higher-spin fields coupled to gravity[8] except that it is more extensive as it includes the helicity-1 case. A physical basis for the inconsistency in the curvature theory can be seen in terms of the famous nonlocalizability of the photon[9] and higher-spin massless particles. There are general arguments, both classical[10] and quantum mechanical,[11] that there is no meaning for the location of a single photon. Penrose and MacCallum argue[10] that an extension of the relativistic center of mass to massless particles of nonzero spin results in a three-dimensional region rather than a one-dimensional world line. The nonlocalizability manifests itself in the curvature theory through the hidden local symmetry. In the gauge $x^0=\lambda$, the hidden local symmetry implies that x^i is a gauge-variant quantity and hence not an observable. Because the photon does not have a well-defined location in spacetime it does not have a well-defined response to a local gravitational field. The nonlocalizability of the photon thus gives a physical picture of the inconsistent coupling of the curvature theory to gravity.

The physical interpretation of the inconsistency suggests that any theory that contains massless particles of spin 1 or greater should exhibit a hidden local symmetry rendering the position variable unobservable. Spinning particles or spinning strings do contain such particles and the hidden symmetry is none other than the local supersymmetry of the world line or world sheet. Gauge fixing only reparametrization invariance leaves a residual symmetry of the local supersymmetry under which the position is gauge variant and so unobservable. Note that the bosonic string does not have spinning massless particles at the classical level and so the classical constraints of bosonic string theory do not exhibit a hidden local symmetry. On the other hand, the simplest spinning point-particle theory contains only a massless spin-$\frac{1}{2}$ particle and yet still exhibits a local supersymmetry. The local supersymmetry here can still be associated with nonlocalizability as chiral spin-$\frac{1}{2}$ particles are also not localizable[11] and one can think of the simplest spinning particle as arising from two chiral spin-$\frac{1}{2}$ particles. Finally, curvature theories[7] with powers of κ different from 1 do not contain massless particles of spin greater than or equal to 1 and do not exhibit a hidden local symmetry. Such theories can be consistently coupled to gravity. These theories have only the two constraints reflecting the reparametrization invariance of the action which is not destroyed by the introduction of a curved metric.

Spinning particles or spinning strings can be coupled consistently to gravity by superspace techniques. There the manifest supersymmetry of the action implies that a point in superspace does have a well-defined response to the metric evaluated at a point in superspace. The superspace action expanded in terms of component fields[12] implies a nonminimal coupling of the Riemann tensor contracted into the internal angular momentum. The minimally coupled theory is inconsistent and the role of the nonminimal term is to restore consistency. The analogy between the hidden local symmetry of the curvature theory and local supersymmetry suggests that the inconsistency of the curvature theory might be overcome by some analog of superspace techniques. However, the inability to understand the hidden symmetry as a local symmetry of the action presents an obstacle to this approach.

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