Flow in Powders: From Discrete Avalanches to Continuous Regime

Jean Rajchenbach
Laboratoire d'Optique de la Matière Condensée, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris CEDEX 05, France
(Received 12 February 1990)

The flow under gravity of a granular material can exhibit either intermittent avalanches or a steady regime. The transition from the discrete to the steady regime is shown experimentally to display hysteresis. Next, a relationship between the current and the slope is established and interpreted.

PACS numbers: 05.40.+j, 46.10.+z, 62.20.−x, 91.60.−x

The dynamical behavior of granular media is of great technological interest, and has given rise to a considerable literature.[1-4] Recently, both experimental and theoretical effort has been devoted to the process of the breaking of equilibrium of a granular pile. Bak, Tang, and Wiesenfeld[5] introduced a model describing the onset of avalanches in a sandpile. In this model, particles are randomly added on a sandpile; the slope of the heap increases until it reaches a critical value at which an avalanche is generated, after which the slope is readjusted to a lower value.

Bak, Tang, and Weisenfeld[5] considered the sandpile as an example of a "self-organized critical system." They computed the dynamics behavior within a cellular automaton model. Kadanoff *et al.*[6] continued the exploration of this model and looked for the most appropriate scaling analysis (finite-size scaling or multifractal analysis) of the size distribution of avalanches versus the size of the system.

Experimentally, avalanches have been studied independently by Jaeger, Liu, and Nagel[7] and by Evesque and Rajchenbach.[8] The same setup was adopted in both cases, which was a rotating cylinder partly filled with powder.[9] For low rotation speeds, the slope increases continuously until an avalanche is generated. The experiences mentioned above[7,8] deal with this regime of intermittent flow, and were in good agreement, in that they did not observe divergencies consistent with self-organized criticality.

For larger rotation speeds, the flow becomes continuous and leads to a steady profile for the free surface. In this paper we focus essentially on the transition between the regime of intermittent avalanches and that of continuous flow.

First, we show the existence of two different critical speeds of rotation according to whether the speed of rotation of the cylinder is increased or decreased. Then, we establish experimentally the relationship between the surface current J of the downward flowing particles and the slope θ in the regime of steady flow.

According to a conjecture of de Gennes[10] and Tang and Bak[11] we show that the current J obeys the following critical law:

$$J \sim (\theta - \theta_c)^m ,$$

where m is determined experimentally. Next, the value of m is interpreted in the frame of a simple model.

Experimental setup.—The experimental setup consists of a hollow Duralumin cylinder (19 cm diameter) with glass faces, which rotates around its horizontal axis at a constant speed Ω. The rotation speed is driven by a quartz clock, and can be varied from 10^{-3} to 60 rpm. The relative accuracy of Ω is of the order of 10^{-3}. The revolving cylinder is partly filled with glass spheres of 0.3 mm diameter. The precision for the diameter of the particles is of the order of 30%.

The choice of the diameter of the grains has been induced by two considerations: first, that the grains must be small compared to the size of the cylinder to avoid boundary effects, and, second, that the particles must be large enough to get rid of electrostatic and humidity effects (which are significant for very fine particles). Care was always taken to avoid the latter effects by using grains that were as dry as possible. With these precautions our measurements of the free-surface profile were reproducible.

The rotating cylinder is a very convenient setup for measuring the dependence of J on θ in the steady regime, because the heap is continuously supplied with new particles upstream and the surface current is exactly known. For a half-filled cylinder the total quantity of matter brought up by the rotation process per unit time is equal to that flowing down; we thus have

$$J(r=0) = \tfrac{1}{2} e \Omega R^2 .$$

Therefore we need only to measure the slope $\theta(r=0)$ of the heap to establish the relationship between J and θ.

Nature of the transition from the discrete to continuous regime of flow.—We observed hysteresis for this transition. We found two different characteristic speeds of rotation Ω_+ and Ω_- according to the sense of variation of Ω (Fig. 1). For perfectly dry particles, we found the Ω_- and Ω_+ are of the order of 0.25 and 0.50 rpm.

We propose the following tentative explanation of this hysteresis. The transition between the discrete and continuous regime of flow occurs when the time t of the fall of a grain becomes comparable to the average delay time T separating two consecutive avalanches,

$$t = T = \langle \theta_{\rm start} - \theta_{\rm stop} \rangle / \Omega .$$

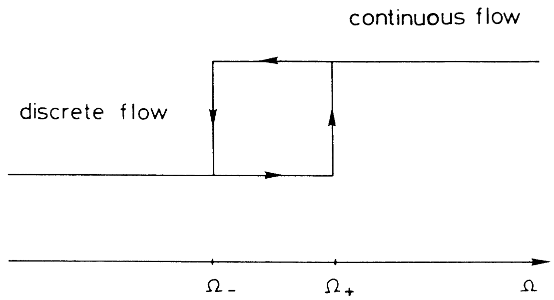


FIG. 1. Within the rotating-cylinder setup, we observe the occurrence of a hysteresis for the transition from the discrete (avalanche) regime of flow to the continuous regime, according the sense of variation of the rotation speed. We attribute this hysteresis to the existence of two characteristic fall times of grains in each regime, which are not the same at the transition.

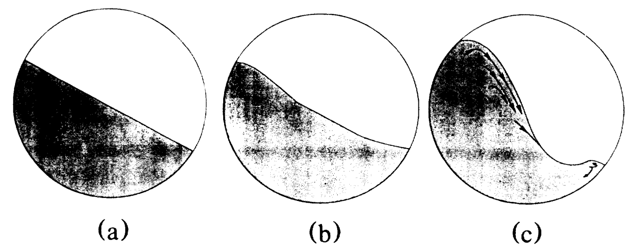


FIG. 2. Profiles of the force surface of the granular material for different rotation speeds of the cylinder. The diameter of the cylinder is 19 cm and the size of the grains is 0.3 ± 0.1 mm. (a) $\Omega = 1.2$ rpm. Just above the transition ($\Omega_+ = 0.5$ rpm) the force surface is nearly a perfect plane. (b) $\Omega = 4.8$ rpm. The force surface exhibits noticeable boundary effects, but they remain confined and measurement of the slope is still possible. (c) For larger rotation speed (here $\Omega = 24$ rpm), we observe that the surface disengages itself from the bulk due to inertial effects.

For the continuous regime, the profile of the surface is steady, and we obtain a characteristic time t_2 for the fall of one particle. On the contrary, for the discrete regime, the profile of the free surface is continuously readjusting itself, and we get another characteristic time t_1 for the fall. At the change of the regime of flow, these two times t_1 and t_2 are not the same. This is, in our opinion, the origin of the observed hysteresis. We deduce therefore

$$\Omega_+ (\langle\theta_{\rm start}\rangle - \langle\theta_{\rm stop}\rangle)/t_1 \,,$$

$$\Omega_- = (\langle\theta_{\rm start}\rangle - \langle\theta_{\rm stop}\rangle)/t_2 \,.$$

Finite-size effects.— In his study of the displacement of a bulldozed sand heap, Bagnold[12] attributed the difference $\langle\theta_{\rm start} - \theta_{\rm stop}\rangle$ to dilatant properties of granular materials: The superficial layer must dilate before entering into motion. For a small diameter of the rotating cylinders, we showed that this mechanism is not relevant. We visualized that the lower wall plays an essential role for the stop of the avalanche and controls entirely the value of $\theta_{\rm stop}$: All the avalanche gets immobilized after the upward propagation of a stopping front which comes from the lower wall and restores a plane shape all along the free surface. In conformity with the translational invariance all along the surface, and as is well accounted for in Coulomb's model[13] of the angle of repose, the angle of start and the thickness h of the sliding layer only depend on purely local conditions. $\langle\theta_{\rm start}\rangle$ is thus independent of the cylinder size. However, as required by conservation of matter, $\langle\theta_{\rm start} - \theta_{\rm stop}\rangle$ has to vary as $2h/R$, according to this stopping process.

Nevertheless, Bagnold's mechanism is expected to become relevant for the case of a large cylinder: Since the upward propagation of the stopping front is associated with a rearrangement and a compression of the packing, a new dilatation is then required to trigger another avalanche. In this last case, $\langle\theta_{\rm start} - \theta_{\rm stop}\rangle$ tends to a constant value, estimated to be 0.4° by Bagnold.[12] Since the time of fall of a grain should become infinite for an infinite cylinder diameter, and following the argument previously developed (see section above) we expect that the intermittent regime of flow would disappear for an infinite system size.

Continuous-flow regime.— In Fig. 2, we show some qualitative changes in the aspect of the surface profile when the speed of rotation is increased. It concerns the case of a rotating cylinder half filled with glass spheres of 0.3 ± 0.1 mm diameter. For rotation just above Ω_+, the surface profile is a nearly perfect straight line, with a slope $\theta = 27°$ [Fig. 2(a)]. Then, for rotation speed in the range 5–12 rpm, the surface profile exhibits noticeable boundary effects, but these remain confined in the vicinity of the wall [Fig. 2(b)]. Next, for increasing rotation speed (from 12 to 25 rpm), the up and down boundary regions merge, and the profile becomes S shaped as previously noticed by Franklin and Johanson[9] and Brown and Richard.[2] For speeds of rotation greater than 25 rpm, one observes a separation of a surface layer from the upper wall and also a backflow near the lower wall [Fig. 2(c)].

Nature of the boundary effects in the continuous regime.— We neglect in all that follows the effect of inertia, which is valid for the limit $\Omega \to 0$. In this case the finite size of the container induces a depletion of grains near the upper wall, and an accumulation near the lower one. For the continuous regime of flow, and for the case of an incompressible medium, de Gennes[10] obtained the profile of the flux of particles on the free surface. He wrote the following equation of conservation for the granular material:

$$\Omega e r + {\rm div} J = 0 \,,$$

with the boundary conditions

$$J(\pm R) = 0 \,,$$

where R and e are the radius and the thickness of the rotating cylinder, and Ωer is the quantity of matter brought to the surface by the rotation process. One easily obtains

$$J = \tfrac{1}{2} e \Omega R^2 (1 - r^2/R^2) .$$

For higher rotation speeds, one must take into account inertial effects. Consider a grain in contact with the upper wall and arriving at the surface of the heap. Two cases have to be considered, according to whether the centrifugal acceleration Ω^2/R is larger or smaller than the radial component of gravity $g \sin\theta$. For $\Omega^2 > g \times (\sin\theta)/R$, the boundary particles remain stuck to the wall by centrifugal force until an angle $\alpha = \arcsin(\Omega^2 R/g)$, which is larger than the steady angle θ of the slope. Beyond this angle α, the grains are launched in parabolic trajectories of length $\lambda = (2\Omega^2 R^2/g)\sin\theta/\cos^2\theta$. This explains the observed separation [Fig. 2(c)] which is seen experimentally to occur for $\Omega = 24$ rpm and is fully consistent with our interpretation.

When $\Omega^2 < g \sin\theta$, the boundary particles no longer adhere to the wall when they are at the free surface. Their trajectories are not exactly parabolic, since there is still contact between particles and losses by friction.

We have no clear explanation of the lower-boundary effects in the case of the continuous regime of flow. For discrete avalanches, we have shown that the range of these effects is infinite. On the other hand, for the continuous regime, matter is progressively evacuated by the rotation process as it accumulates, thus preventing an infinite expansion of lower-wall effects.

Relationship between the surface current and the slope of the heap.—The measured scale of variation of the current J is 50, since below $\Omega_- = 0.25$ rpm the current is intermittent, and above $\Omega = 12$ rpm, we observe the merging of the up and down boundary regions. In the range $\Omega_+ = 0.50$ rpm to $\Omega = 12$ rpm, within the experimental accuracy, measurements of slope are independent of the sense of variation of the rotation speed.

Our experimental determination (Fig. 3) of the law $J \sim (\theta - \theta_c)^m$ leads to

$$m = 0.5 \pm 0.1 ,$$

which differs from the numerical prediction ($m = 0.7$) of Tang and Bak[11] appropriate for surface flows.

Since our measurements of θ are well away from the regions sensitive to boundary effects, they deal with a steady regime of flow where particles have attained their limiting velocity.

Physical meaning of the index m.—We propose to relate the value of the exponent m to the dissipation process occurring in the granular flow.

For the sake of simplicity, consider the flow of a fluid of thickness h on an inclined plane. For a Newtonian fluid of viscosity η, the current is given by

$$J = (\rho g h^3/3\eta)\sin\theta .$$

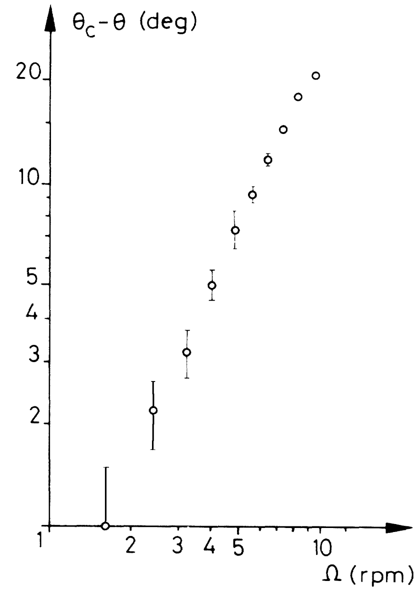


FIG. 3. A log-log plot of $\theta - \theta_c$ vs Ω (Ω is proportional to the current J). We find $\Omega \sim (\theta - \theta_c)^m$, with $m = 0.5 \pm 0.1$.

Since the equilibrium angle θ_c is zero in this case, we obtain

$$m = 1 .$$

In such Newtonian fluids, the mechanism which is assumed to be at the origin of the loss of momentum arises from collisions due to random Brownian motion. Thus one can properly define viscosity when the mean Brownian velocity greatly exceeds the average speed v. In the case of non-Brownian particles, we propose an elementary approach based upon Coulomb's balance of stress available for statics, to which we add a dissipation term due to interparticle collisions.

As conjectured by Bagnold[1] for non-Brownian particles, both the momentum lost at each collision and the rate of collision per unit time are proportional to ∇v, so that the friction force varies like $\alpha(\nabla v)^2$. Consequently, equality between dissipative stress and motive stress leads in the steady regime to

$$-\alpha(\partial v/\partial z)^2 + \rho g z (\sin\theta - \mu\cos\theta) = 0 ,$$

where $\mu = \tan\theta_c$ is the solid friction coefficient of Coulomb's theory,[13] and $0z$ is the axis normal to the flow and oriented downward.

In the last equation, we consider that ρ and μ varied weakly with v in the vicinity of the equilibrium. A limited expansion of this equation near $\theta = \theta_c$ and successive integrations lead to

$$v(z) = \tfrac{2}{3} [\rho g h^3/\alpha(\cos\theta_c)]^{1/2} [1 - (z/h)^{3/2}] (\theta - \theta_c)^{1/2} ,$$

$$J = \tfrac{2}{5} [\rho g h^5/\alpha(\cos\theta_c)]^{1/2} (\theta - \theta_c)^{1/2} .$$

An essential point is that the power law

$$J \sim (\theta - \theta_c)^{1/2}$$

does not depend on the detailed geometry of the container. It proceeds only from Bagnold's description of dissipation and on the expansion of the condition of stresses equilibrium near θ_c. It can be generalized to other geometries of flow such as the rotating cylinder, and is consistent with our experimental determination.

I thank P. G. de Gennes for stimulating and enlightening discussions. I thank J. Duran, S. Savage, M. Fermigier, and A. Mehta for encouragement and a critical reading of the manuscript. Laboratoire d'Optique de la Matière Condensée is unité associée No. 800 du Centre National de la Recherche Scientifique.

[1]R. A. Bagnold, *The Physics of Blown Sand and Desert Dunes* (Methuen, London, 1941); Proc. Roy. Soc. London A **225**, 49 (1954).

[2]R. L. Brown and J. C. Richard, *Principles of Powder Mechanisms* (Pergamon, New York, 1966).

[3]S. B. Savage, J. Fluid Mech. **92**, 53 (1979); S. B. Savage and K. Hutter, J. Fluid Mech. **199**, 177 (1989), and references therein.

[4]P. C. Johnson and R. Jackson, J. Fluid Mech. **176**, 67 (1987).

[5]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[6]L. P. Kadanoff, S. R. Nagel, L. Wu, and S. M. Zhou, Phys. Rev. A **39**, 6524 (1989).

[7]H. M. Jaeger, C. H. Liu, and S. R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).

[8]P. Evesque and J. Rajchenbach, C. R. Acad. Sci. (Paris), Ser. 2, **307**, 223 (1988).

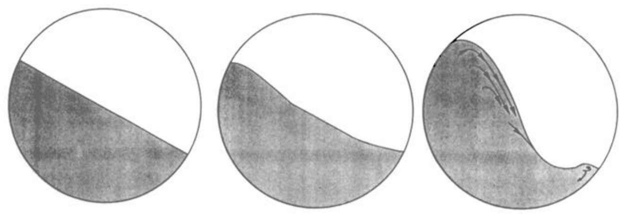
[9]F. C. Franklin and L. N. Johanson, Chem. Eng. Sci. **4**, 119 (1935).

[10]P. G. de Gennes (to be published).

[11]C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988).

[12]R. A. Bagnold, Proc. Roy. Soc. London A **295**, 219 (1966).

[13]C. Coulomb, in *Mémoires de Mathématiques et de Physique présentés à l'Académie des Sciences par divers savants et lus dans les assemblées, année 1773* (Paris, 1776), p. 343.



(a) (b) (c)

FIG. 2. Profiles of the force surface of the granular material for different rotation speeds of the cylinder. The diameter of the cylinder is 19 cm and the size of the grains is 0.3 ± 0.1 mm. (a) $\Omega = 1.2$ rpm. Just above the transition ($\Omega_+ = 0.5$ rpm) the force surface is nearly a perfect plane. (b) $\Omega = 4.8$ rpm. The force surface exhibits noticeable boundary effects, but they remain confined and measurement of the slope is still possible. (c) For larger rotation speed (here $\Omega = 24$ rpm), we observe that the surface disengages itself from the bulk due to inertial effects.