Quantum Hall Effect and the Relative Index for Projections

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We define the relative index, Index(P,Q), for a pair of infinite-dimensional projections on a Hilbert space to be the integer that is the natural generalization of $\dim(P)-\dim(Q)$ in finite-dimensional vector spaces. We show that the Hall conductance for independent electrons in the plane is the relative index where P and Q project on the states below the Fermi energy for Hamiltonians that differ by a quantum flux and the Fermi energy is appropriately placed. This approach is closely related to, and sheds light on, Bellissard's interpretation of the Hall conductance as an index.

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The physical conditions for the integer quantum Hall effect were identified by Laughlin.[1] Since then, two mathematical frameworks that identify the integers with topological invariants have been developed: The first one,[2] which has its roots in a work of Thouless, Kohmoto, Nightingale, and den-Nijs, identifies the Hall conductance with a Chern number associated to the adiabatic curvature. The second, due to Bellissard,[3] identifies it with the index of a certain operator that arises in Connes theory.[4] As both frameworks are differential geometric in character and ultimately reduce to redressing Kubo's formula, it has been felt that they are related, and the original motivation for our work was indeed to examine this relation.

One of the things that makes such a comparison difficult is the formidable machinery employed by Bellissard and the *Deus ex machina* role played by Connes theory of noncommutative differential geometry.[4] For this reason we felt that it would be desirable to rederive Bellissard's results in a way that, though rigorous, is motivated by intuition about the Hall effect rather than operator algebra and noncommutative differential geometry. The outline of such a derivation is presented below.

For reasons of space we shall not present a detailed comparison of the Chern versus the index approach, and rather state what we believe is one important conclusion, namely, that they are complementary: The Chern number approach is the natural one for closed, multiply connected multiparticle systems, while the index approach captures important features of open, simply connected and (up to now) single-electron systems. Further details shall be given elsewhere.[5]

The approach we describe below is motivated by the following view of the Hall conductance: Let an infinitesimally thin flux tube pierce the plane at the origin. The Hall conductance is the amount of charge transported to infinity as the flux increases adiabatically by a unit of flux quantum. For the case of the full lowest Landau level, an explicit calculation described, for example, by Laughlin[1] shows that one state is lost in accordance with the unit Hall conductance of a full Landau level. (If, instead, a unit of flux is removed, an extra unit of charge is "sucked in" from infinity causing a charge near the flux tube to be pushed to the next Landau level.)

The essence of this view is the comparison of two infinite-dimensional projections: Let P denote the spectral projection on the states below the Fermi energy for the initial Hamiltonian and Q that for the final Hamiltonian. $\mathrm{Tr}P=\mathrm{Tr}Q=\infty$, of course, as both count the infinitely many electrons below the Fermi energy. In finite-dimensional vector spaces

$$\mathrm{Tr}(P-Q)=\dim(P)-\dim(Q) \qquad (1)$$

counts the difference in dimensions, and in the case at hand the analog of that should count the number of charges transported to (and from) infinity as the flux increases by a quantum unit.

Here we shall describe a natural generalization of Eq. (1) to Hilbert space (infinite-dimensional, orthogonal) projections so that $P-Q$ is compact. ($P-Q$ is compact, if its eigenvalues $\lambda_n\to 0$ as $n\to\infty$.[6]) We shall call the relative index, denoted by Index(P,Q), the *integer* which is the infinite-dimensional analog of $\dim(P)-\dim(Q)$. The basic features of the relative index are described below. We outline the proof that in the context of the quantum Hall effect $(P-Q)^3$ is trace class[5] (i.e., that $\sum|\lambda_n|^3<\infty$) provided that the Fermi energy lies in a gap, and recall Bellissard's observation that the *analog* of the left-hand side of Eq. (1) is Kubo's formula in disguise. The paper concludes with observations regarding how this framework could accommodate Laughlin's theory of the fractional Hall effect.

Basic to our definition of Index(P,Q) is the observation that the spectrum of $P-Q$ within $(-1,1)$ is symmetric with respect to zero. (There is no spectrum outside $[-1,1]$, of course.) That is, if $-1<\lambda<1$ is an eigenvalue of $P-Q$ with multiplicity $m(\lambda)$ then $-\lambda$ is also an eigenvalue with the same multiplicity. To see this, note that by a simple computation

$$(P-Q)(P_\perp-Q)=-(P_\perp-Q)(P-Q)\,, \tag{2}$$

$$(P-Q)^2+(P_\perp-Q)^2=1\,, \tag{3}$$

where $P_\perp \equiv 1-P$. It follows from (2) that $P_\perp-Q$ maps the eigenspace of λ to that of $-\lambda$. From (3), which is originally due to Kato,[7] it follows that $(P_\perp-Q)^2$ is $1-\lambda^2$ on the eigenspace with fixed λ and so $P_\perp-Q$ is invertible if $\lambda^2\neq\pm 1$, thus establishing our claim.

Armed with this, we define

$$\begin{aligned}\mathrm{Index}(P,Q)&\equiv m(1)-m(-1)\\ &=\dim(\mathrm{Range}P\cap\mathrm{Range}Q\perp)\\ &\quad-\dim(\mathrm{Range}Q\cap\mathrm{Range}P\perp)\\ &=\mathrm{Index}(QP)\,,\end{aligned} \tag{4}$$

where the last expression is the index of a map from the range of P to the range of Q.

Suppose now that $(P-Q)^{2n+1}$ is trace class, i.e., $\sum_j|\lambda_j|^{2n+1}<\infty$, then the analog of Eq. (1) is

$$\mathrm{Index}(P,Q)=\mathrm{Tr}(P-Q)^{2n+1}\,. \tag{5}$$

Incidentally, so far as we know, this gives the first proof that for infinite-dimensional P and Q, if $(P-Q)^{2n+1}$ is trace class then the trace is an integer, a fact that appears to be new even for $n=0$.

Equation (5) also shows that the right-hand side is independent of n provided n is large enough. This is the Hilbert-space analog of the identity $\mathrm{Tr}(P-Q)=\mathrm{Tr}(P-Q)^{2n+1}$ which holds in finite-dimensional vector spaces by the cyclicity of the trace (which we do not have).

Because the spectrum is unitary invariant we clearly have

$$\mathrm{Index}(P,Q)=\mathrm{Index}(UPU^\dagger,UQU^\dagger)\,, \tag{6}$$

with U unitary. In particular, it is gauge invariant.

Another fact is "linearity," that is, if $P-Q$ and $Q-R$ are compact, then

$$\mathrm{Index}(P,Q)=\mathrm{Index}(Q,R)+\mathrm{Index}(R,P)\,. \tag{7}$$

This relation is a trivial consequence of Eq. (5) if the differences are trace class (n can be chosen 0), but by a more complicated argument[5] linearity holds for compacts. This is important for the application where the differences are, in fact, *not* trace class, in the interesting cases.

In the context of the quantum Hall effect we are interested in spectral projections P and Q that are related by a unitary gauge transformation U associated with a flux tube carrying a unit of quantum flux. For example, take for U the multiplication operator given by the unimodular function $u(z)=z/|z|$ with $z\equiv x+iy$ and $Q\equiv UPU^\dagger$. Define

$$\begin{aligned}\mathrm{Charge}(P,U)&\equiv\mathrm{Index}(P,UPU^\dagger)\\ &=\mathrm{Tr}([P,U]U^\dagger)^{2n+1}=\mathrm{Index}(PUP)\,,\end{aligned} \tag{8}$$

provided n is such that $(P-Q)^{2n+1}$ is trace class. The first identity is simple, and the second is not.[4,5] We shall not make use of it here except to note that this is the Bellissard-Connes index.

Combining Eqs. (6)-(8) we get

$$\mathrm{Charge}(P,U^n)=n\,\mathrm{Charge}(P,U)\,, \tag{9}$$

which has the interpretation that n units of quantum flux transport n times the charges transported by a single quantum flux.

The basic tool to determine what power, if any, of $P-Q$ is trace class is Russo's theorem,[8] which (specialized to integer powers and self-adjoint operators) says that A^p is trace class, $p>1$, if its integral kernel $A(z,z')$ satisfies

$$\int\left(\int|A(\mathbf{x},\mathbf{y})|^{p/(p-1)}d\mathbf{x}\right)^{p-1}d\mathbf{y}<\infty\,. \tag{10}$$

Taking U to be the multiplication by the unimodular $u(z)$, the integral kernel of $P-Q$ is

$$P(z,z')[1-u(z)/u(z')]\,. \tag{11}$$

For $u(z)=z/|z|$,

$$|u(z)-u(z')|\le|z-z'|[\mathrm{Min}(|z|,|z'|)]^{-1}\,, \tag{12}$$

so the integral kernel decays essentially like $1/|z|$ near the diagonal. Now provided there is decay away from the diagonal, and in particular if

$$P(z,z')\le C(1+|z-z'|)^{-3-\epsilon}\,, \tag{13}$$

it follows essentially by power counting that $(P-Q)^3$ is trace class.

It follows from Eq. (11) that if $P-Q$ is trace class and if P has a sufficiently smooth integral kernel and U a gauge transformation then Charge(P,U) vanishes. (The integral kernel of $P-Q$ vanishes on the diagonal.) We therefore learn that in this framework a nontrivial integer in the quantum Hall effect results from projections that differ by a compact but non-trace-class operator. This requires that the system be "open" because, for Schrödinger operators associated with a finite box, and Dirichlet boundary conditions, P is automatically finite dimensional, and the index is zero. The vanishing of the relative index reflects the fact that the charge cannot leave the box. This is the sense in which the framework describes open systems.

It is clear that Charge(P,U) is invariant under deformations of P as long as $[P,U]$ remains compact, and, in particular, as long as P satisfies Eq. (13). It follows that between Hall plateaus, P must develop long tails. It is instructive to compare this with what happens in the Chern number framework. The latter gives nontrivial integers already for finite (but multiply connected) systems, where the basic mechanism that allows the Hall conductance to change is level crossings rather than loss of compactness. Kunz[2] and Niu and Thouless[2] examined this framework in the thermodynamic limit, and argued that P develops long tails between plateaus.

The condition in Eq. (13) can be shown to follow from the fact that the Fermi energy lies in a gap.[5] If the Fermi energy lies in the region of localized states then Eq. (13) holds by results of Fröhlich and Spencer[9] in the absence of magnetic field. The common wisdom suggest that Eq. (13) holds also with magnetic fields.

It is natural to ask what is the direct interpretation of $\mathrm{Tr}(P-Q)^3$. Explicitly it is given by

$$\int dz_1\,dz_2\,dz_3\,P(z_1,z_2)P(z_2,z_3)P(z_3,z_1)\left(1-\frac{u(z_1)}{u(z_2)}\right)\left(1-\frac{u(z_2)}{u(z_3)}\right)\left(1-\frac{u(z_3)}{u(z_1)}\right). \tag{14}$$

It is an observation due to Bellissard that if one takes $u(z)=z/|z|$ and if the Schrödinger operator is invariant under magnetic translations (and if the integral kernel of P has enough decay), then the integral over the u's in Eq. (14) can be carried out explicitly by a nontrivial formula of Connes. The resulting equation can be shown to reduce to Kubo's formula for the conductance, on the one hand, and to the Connes index, Index(PUP), on the other hand. (Actually, Bellissard shows a more general result that allows for a random background potential.) The derivation of this result, as well as its extension to situations without translation invariance, shall be discussed in Ref. 5. For our purpose it is enough to note that there is this link between Eq. (5) and Kubo's formula.

We now conclude with remarks about the fractional Hall effect. In the Chern number approach (which is intrinsically a many-body theory) fractions arise from a degeneracy of the ground state. Plateaus for fractions then imply that these degeneracies are stable, which is not the generic situation. Various authors[10,11] considered mechanisms that could give such stability. On the other hand, in Laughlin wave-function theory it appears that, depending on the choice of the underlying two-dimensional manifold, the ground state can be either degenerate or simple. As a consequence, some authors have expressed the opinion that degeneracies are not an essential ingredient of the fractional effect.[1]

The framework described above is intrinsically a one-particle theory, for we have taken P to be the spectral projection on the one-particle states below the Fermi energy. As such, it cannot accommodate the fractional effect without some serious upgrading. In particular, we do not know what replaces P in a many-body theory. Nevertheless, the framework suggests what we believe is an interesting mechanism to get fractions, that avoids degeneracies, and appears to be in agreement with the basic ideas in Laughlin.[1] We have taken the Hall conductance to be Charge(P,U), where U is associated with a single quantum flux. By Eq. (9) we could just as well have taken the Hall conductance to be Charge$(P,U^3)/3$. Of course, if $(P-Q)^3$ is trace class, this is still an integer because the numerator is a multiple of 3. However, it is possible for Charge$(P,U^3)=1$, with P a spectral projection and U unitary in situations where $[P,U]$ has bad decay properties while $[P,U^3]$ has good decay properties. A simple (one-body) example where this is the case is to take P to be the spectral projection associated to the lowest Landau level and U the unitary $(z/|z|)^{1/3}$, choosing a branch for the cube root. One can imagine that in the fractional Hall effect, the many-body analog of Charge(P,U) would be such that Charge$(P,U^3)=1$ with U associated with the quantum flux. The conductance would then be $\frac{1}{3}$. For this to happen, it must be that the analog of $[P,U]$ would have long-range correlation but not $[P,U^3]$. This appears to be supported by the ideas and calculations of Girvin and MacDonald,[12] and Read[12] about the off-diagonal long-range order in the Laughlin state. This could then lead to fractions without appeal to degeneracies.

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