Statistical Mechanics of Euler Equations in Two Dimensions

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We formulate the statistical mechanics of a two-dimensional inviscid incompressible fluid in a manner which, for the first time, respects all conservation laws. For a special case, we demonstrate that a mean-field theory is exact. A consequence of our arguments is that, in an inviscid fluid evolving from initial conditions to statistical equilibrium, only the energy and certain one-body integrals appear to be conserved. Our methods may be applied to a variety of Hamiltonian systems possessing an infinite number of conservation laws.

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Kirchoff[1] observed that the equations of motion for point vortices in a two-dimensional, inviscid, incomprehensible[2] fluid can be derived from a Hamiltonian:

$$\mathcal{H}=-\sum_{i\neq j}\omega_i\omega_j\ln|\mathbf{r}_i-\mathbf{r}_j|,\quad \omega_i\frac{d\mathbf{r}_i}{dt}=\nabla_i\times\mathcal{H}. \tag{1}$$

The conjugate variables are the coordinates of the ith vortex x_i,y_i and we use the notation

$$\mathbf{u}=\nabla\times\psi=(\partial_y\psi,-\partial_x\psi),\quad \omega=\nabla\times\mathbf{u}=-\nabla^2\psi, \tag{2}$$

where ω is the (scalar) vorticity field, $\mathbf{u}$ is the velocity field, and ψ is the stream function. A substantial body of work is based on the premise that the properties of this system, the point vortex gas, have implications for the flow described by the Euler equations.[3] In particular, given a Hamiltonian it is natural to ask about equilibrium properties, using the methods of statistical mechanics.

The evolution of large-scale coherent structures (or blobs) is an oft-noted feature of two-dimensional fluid flow. The notion that blobs might be a simple equilibrium phenomenon was suggested by Onsager.[4] He pointed out that in bounded regions and at high energies the vortex gas, with Hamiltonian (1) and all $|\omega_i|=\omega_0$, gives rise to clusters of vortices of the same sign. Onsager argued that the bounded phase space implies that above a certain energy the number of states available to the gas decreases as a function of energy, giving rise, at least for a finite number of vortices, to "negative temperature" states.

While interest in this system has surfaced on many occasions since Onsager's proposal,[5] unresolved problems remain. For example, questions have been raised as to whether negative temperatures and blobs persist in the thermodynamic limit.[6] Onsager himself was uncertain how the statistics of point vortices applied to the more familiar situation in which initial conditions specify a continuous distribution of vorticity. A related issue concerns the proper treatment of the infinite number of integrals of motion in two-dimensional Euler flow: $\int_\Omega d^2r\,\omega^n(\mathbf{r})$ for integer n, where Ω is the region containing the fluid. These quantities are conserved since

$$\frac{d}{dt}\int_\Omega d^2r\,\omega^n(\mathbf{r})=\int_\Omega d^2r\,n\omega^{n-1}(\mathbf{r})\left[\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega\right]=0. \tag{3}$$

For $n=2$, this integral is known as the enstrophy.

The point vortex model represents a singular case of Euler flow, since constants of motion with $n>1$ involve powers of delta functions. A natural way to go about eliminating this defect is to write down a partition function incorporating the constraints as is usual in statistical mechanics:

$$\int\mathcal{D}\psi\exp\int_\Omega d^2r\left[-\frac{1}{T}(\nabla\psi)^2+\sum_n\alpha_n\omega^n(\mathbf{r})\right], \tag{4}$$

where the constants α_n, $n\geq 1$, and $1/T$ are Lagrange multipliers. In taking this approach, Kraichnan discarded all constants of motion except for the energy and the enstrophy;[7] however, integrals of other powers of the vorticity cannot be neglected in the study of long-wavelength properties of Euler flow in a compact domain.

In this paper we construct a theory of statistical equilibrium for the two-dimensional Euler fluid which incorporates all constants of motion. The equilibria generically feature blobs. We find that in a fluid evolving from some initial condition to statistical equilibrium, only the energy and integrals linear in the vorticity appear to be conserved. All other constants, including the enstrophy, are found to be altered. This situation reflects the fact that averages of such quantities over a finite area need not coincide with their unaveraged values.

A symmetry of the Euler equations enables us to include all constants of motion in our statistical mechanics. The invariance of physical quantities under smooth area-preserving coordinate reparametrizations (the group of area-preserving diffeomorphisms) leads, in two dimensions, to local conservation of vorticity.[8] Integrals over Ω of any smooth function of the vorticity are conserved by the flow; these quantities are the Casimirs of the theory.[9] Equivalently, we may say that the vorticity distribution function $G(\omega)$, which yields the measure of the subset of Ω on which the vorticity takes on a value less than ω, is preserved by the flow.

The preceding assertions follow from the Hamiltonian formulation of the Euler equations described by a number of authors.[8-11] Our equilibrium statistical mechanics will be obtained by averaging over all configurations of the fluid which share the same $G(\omega)$ and energy,[12] with a weighting arising naturally from the Hamiltonian.[13] As in many applications of statistical mechanics, we cannot rigorously justify our assumption of ergodicity.[14]

We now sketch the construction of our theory. For ease of presentation we consider only the very simplest case: two-dimensional Euler flow in a disk Ω of radius 1. We impose free boundary conditions so that the only role of the boundary is to make the volume finite; consequently, we take $-(1/2\pi)\ln|\mathbf{r}-\mathbf{r}'|$ as our Green's function. We also require that our vorticity distribution $G(\omega)$ be such that the magnitude of the vorticity is bounded by some $|\omega|_{\max}$.

Our Hamiltonian takes the form[10,11] $\mathcal{H}=\frac{1}{2}\int_\Omega d^2r \times \mathbf{u}^2(\mathbf{r})$. This non-negative quantity is the kinetic energy of the fluid once we rescale $\mathbf{r}$ to be dimensionless and set the density to 1. We integrate by parts and ignore the contribution of the boundary to obtain

$$\mathcal{H}=-\frac{1}{4\pi}\int_\Omega d^2r\int_\Omega d^2r'\,\omega(\mathbf{r})\omega(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'|\,. \tag{5}$$

We next write down a canonical partition function: $\int \mathcal{D}^g\omega \exp[-\mathcal{H}(\omega)/T]$. The superscript g refers to the fact that we integrate over configurations which have a given vorticity density function $g(\omega)=dG(\omega)/d\omega$. For purposes of counting states, we need to regularize our functional integral. We do so by incorporating a lattice spacing, a. Our Hamiltonian becomes

$$\mathcal{H}^a=-\frac{a^4}{4\pi}\sum_{i\neq j}\omega_i\omega_j\ln|\mathbf{r}_i-\mathbf{r}_j|+(\text{self-energy})\,, \tag{6}$$

where the i,j take values on a lattice of side a in the region Ω, and the ω_i are averages of $\omega(\mathbf{r})$ over lattice boxes of side a. The total self-energy scales as $a^2\ln(a^2)$ and so its contribution to the Hamiltonian vanishes as $a\to 0$. Up to a factor of a^4, our regularized Hamiltonian looks like that of the point vortex gas (1), but it is distinguished by the underlying lattice, which is required in order to impose the conservation laws.

To understand the effect of the regularization of the partition function on the functional integration, it is easiest to consider an example. Take $g(\omega)$ to have the form $(\pi-\alpha)\delta(\omega)+\alpha\delta(\omega-1)$, where π is the area of Ω. That is, $G(\omega)$ describes a vorticity distribution with the property that the area upon which the vorticity takes the value 1 is α; the vorticity vanishes elsewhere. Then with lattice spacing a we obtain $N=\pi/a^2$ lattice points, upon which the vorticity takes value 1 on α/a^2 points and value 0 on the remaining points. The functional integration varies the vorticity field over all possible ways of allocating the α/a^2 1's and $N-\alpha/a^2$ 0's among the N lattice sites, with each site occupied by exactly one 1 or 0. It is clear that $G(\omega)$ is approached exactly as $a\to 0$. The limiting process, in which $N\to\infty$ at constant total system volume and the distance of closest approach of two vortices $a\to 0$ at the same rate, distinguishes our system from the point vortex gas. For a continuous $G(\omega)$, we slice the range of the vorticity field into intervals, and choose the relative numbers of lattice sites on which the vorticity falls within a given interval, so as to converge to $G(\omega)$ in the limit of vanishing lattice spacing.

We now outline our argument that the partition function converges to a well-defined and nontrivial limit as the lattice spacing vanishes. In fact, we can derive an explicit condition which the equilbria must satisfy. The reason we can do so is that, for a certain class of vorticity distributions, we can prove that a mean-field theory is exact, as one might anticipate from the long-range nature of the interaction. This class of vorticity distributions consists of those for which $|\omega|_{\max}$ is finite.

The validity of the mean-field theory is a consequence of four factors: (i) the strong constraint imposed by the conservation of a $G(\omega)$ of this type; (ii) the independence of the range of the potential on the lattice spacing a; (iii) the smoothness of the potential away from the source; and (iv) the mild divergence of the potential at the source.

Our proof divides into two steps. The first step is to argue that given a $G(\omega)$, we can approximate the energy to within accuracy ε by considering only structure above a fixed length scale l. We obtain the Hamiltonian $\mathcal{H}^l$ given by (6) but with l replacing a and the ω_i now averages over boxes of side l. The scale l is determined by $|\omega|_{\max}$ and ε; we allow ε and l to vanish at the end of the calculation. The $\mathcal{H}^l$ approximates the energy to the desired accuracy *uniformly* over the set of configurations allowed by $G(\omega)$. A consequence of uniform convergence is that we do not care about correlations on scales smaller than l, so long as we satisfy the constraints imposed by $G(\omega)$. We take $l\to 0$ at the end of the calculation. For notational simplicity we shall not write this limit explicitly.

Our second step is to calculate the entropy S of a system with a given vorticity field and lattice cutoff a by regarding lattice points within a distance l of each other as independent. The entropy is dominated by the large number of isoenergetic configurations of the $(l/a)^2$ vortices, and may be explicitly calculated.

We may view $g(\sigma)$ as determining the total number of squares of side a on which the vorticity takes a value very close to σ. We define the quantity $\rho(\sigma,\mathbf{r})$ as the density of squares of vorticity σ within a distance l of $\mathbf{r}$.

ρ must satisfy two conditions: (c1) $\int_{-\infty}^{\infty} d\sigma \rho(\sigma,\mathbf{r})=1$, which enforces incompressibility; and (c2) $\int_\Omega d^2 r\, \rho(\sigma,\mathbf{r})=g(\sigma)$, which correctly normalizes the density. Note that $\omega(\mathbf{r})$, the vorticity density, is given by $\int_{-\infty}^{\infty} d\sigma\, \sigma \rho(\sigma,\mathbf{r})$.

Now we can write the partition function in terms of ρ:

$$\int \mathcal{D}\rho \exp\{-a^{-2}[\mathcal{H}(\rho)/\bar{T}-\bar{S}(\rho,g)]\}\,. \tag{7}$$

Here $\bar{T}=a^{-2}T$ and $S(\rho,g)=a^{-2}\bar{S}(\rho,g)$ is the logarithm of the number of ways of generating ρ, given a vorticity density function g regularized with cutoff a. Since we may regard the $(l/a)^2$ vortex squares of side a which lie in a box of side l around $\mathbf{r}$, and which yield $\rho(\sigma,\mathbf{r})$, as uncorrelated, we obtain the entropy of an ideal gas:

$$\bar{S}(\rho,g)=-\int_\Omega d^2 r \int_{-\infty}^{\infty} d\sigma \rho(\sigma,\mathbf{r}) \ln \rho(\sigma,\mathbf{r}) \tag{8}$$

which does not depend on a. The quantity in square brackets in (7) then does not depend on a, and in the limit of vanishing lattice spacing the integral is concentrated where this quantity is minimized. Stationary points of this quantity occur at

$$\rho(\sigma,\mathbf{r})=\exp\left[-\frac{\sigma}{\bar{T}}\psi(\mathbf{r})+\mu(\sigma)\right]\left[\int_{-\infty}^{\infty} d\sigma' \exp\left[-\frac{\sigma'}{\bar{T}}\psi(\mathbf{r})+\mu(\sigma')\right]\right]^{-1}, \tag{9}$$

where $\mu(\sigma)$ are Lagrange multipliers implicity defined by the constraints (c2). Using (2) we see that minima of (7) correspond to minima of the free energy:

$$\mathcal{F}^g(\psi)\equiv\int d^2 r\left[\frac{(\nabla\psi)^2}{2}+\bar{T}\ln\int_{-\infty}^{\infty} d\sigma \exp\left[-\frac{\sigma}{\bar{T}}\psi(\mathbf{r})+\mu(\sigma)\right]\right], \tag{10}$$

where ψ must satisfy the boundary conditions. Equation (10) is the free energy for a generalized Ising model with logarithmic interactions; it can be independently derived from the Hamiltonian for the two-dimensional Euler fluid using a Kac-Hubbard-Stratanovitch transformation,[15] where the constraints are imposed upon the Hamiltonian by Lagrange multipliers.

We can draw several conclusions from our argument. (1) It was necessary to require that T vanish along with the lattice spacing. It is $\bar{T}$ which determines the energy. (2) It is easy to see that for a neutral system where g is symmetric about the origin, $\psi\equiv 0$ minimizes $\mathcal{F}^g(\psi)$ for $\bar{T}\geq 0$. There are no nontrivial solutions with nonnegative $\bar{T}$ in this case. (3) In general, the vorticity density function g_d derived from the $\omega(\mathbf{r})$ which yields the above minimum is not the same as g. We know that

$$\int_\Omega d^2 r\, \omega(\mathbf{r})=\int_{-\infty}^{\infty} d\sigma\, \sigma g(\sigma)\,, \tag{11}$$

but no other moment of the vorticity is necessarily the same for both g and g_d.

Put another way, suppose our fluid evolves from smooth initial conditions with vorticity distribution G_b, a "bare" vorticity distribution. The evolving flow is stretched and folded, a process which effectively disperses the smoothly distributed vorticity into smaller and smaller scales. Asymptotically in time t, a measurement on scales large compared to the arbitrarily small scales into which the vorticity is dispersed will yield a distribution $G_d(t)$ which will converge to G_d, the "dressed" vorticity distribution, as $t\to\infty$. Since G_d measures averages, it need not coincide with G_b. The energy and one-body integrals are conserved, since they are long-wavelength properties.

Although G_d in general differs from G_b, a trivial consequence of our arguments is that, at a given energy, G_d yields the same equilibrium solution as G_b. Furthermore, the given energy turns out to be precisely the maximum energy compatible with G_d. It follows that the configuration would be dynamically stable.[10] In other words, if we consider the process of solving Eq. (10) to obtain the dressed distribution from a given bare distribution and energy as a mapping, then G_d is a zero-temperature fixed point of the mapping. A physical implication of this result, which we call the "dressed vorticity corollary," is that for a fluid *in statistical equilibrium*, coarse-grained quantities suffice to determine the equilibrium. This observation suggests that our equilibria might persist in the presence of a viscosity acting to smear the small scales. An equivalent way of stating our result is that the long-time dynamics of an inviscid fluid will evolve to a configuration which is a *global* extremum of the energy, subject to satisfying the long-time (dressed) vorticity distribution.

We work out a simple example to show the relation of our work to previous results. We use the same form of the distribution G as in our example of functional integration, and consider for convenience Dirichlet boundary conditions on a disk, with $\bar{T}<0$. We find from (9)

$$\omega^{\alpha,\bar{T}}(\mathbf{r})=-\nabla^2\psi=\frac{\alpha\exp[-\psi/\bar{T}+\mu_1(\alpha,\bar{T})]}{(\pi-\alpha)\exp\mu_0(\alpha,\bar{T})+\alpha\exp[-\psi/\bar{T}+\mu_1(\alpha,\bar{T})]}, \tag{12}$$

where the chemical potentials $\mu_{0,1}(\alpha,\bar{T})$ are used to enforce (c2). This equation describes the statistical equilibrium of an inviscid fluid with our specified vorticity distribution and temperature $\bar{T}$, and is new in this context. A related, but distinct, equation has been derived by a number of previous authors for point vortices using a mean-field argument.[5] Their equation is a special case of ours, as we can see by taking the limit $\alpha \rightarrow 0$, at the same time scaling the vorticity so as to keep the total circulation constant. Fixing $\bar{T}$ less than $-1/8\pi$, the mean-field collapse temperature for point vortices,[5] we obtain

$$\hat{\omega}^{\bar{T}}(\mathbf{r}) = -\nabla^2 \psi$$
$$= \exp(-\bar{\psi}/\bar{T}) \left(\int_\Omega d^2 r \exp(-\bar{\psi}/\bar{T}) \right)^{-1}. \quad (13)$$

Here $\hat{\omega}^{\bar{T}}(\mathbf{r})$ denotes the normalized density of points. For $0 \geq \bar{T} > -1/8\pi$, the solutions of (13) collapse to a point[5] in contrast to the solutions of Eq. (12), which remain continuous and finite.[16]

We remark that conservation laws and fields linear in $\omega(\mathbf{r})$ do not affect our formulation, which we expect to be applicable to a wide variety of Hamiltonian systems possessing infinite families of Casimirs, among them many of those described in Ref. 9. In particular, it may be relevant to the two-dimensional guiding-center plasma.[5]

More generally, a G with $|\omega|_{\max}$ unbounded may be physically relevant. Our mean-field argument may fail in this case, because $\bar{S}$ diverges as a vanishes, the energy no longer necessarily converges uniformly in l, and/or the self-energy contribution can no longer be ignored. These considerations lead us to expect that G exists such that T is finite as the lattice spacing $a \rightarrow 0$. Such regimes are of interest because we could couple them to thermal (e.g., molecular) degrees of freedom.

We learned after this manuscript was submitted that Eq. (12) and its generalizations had been derived earlier by Lynden-Bell in the context of stellar dynamics.[17] There the particles interact by a gravitational potential. The equation of motion is the collisionless Boltzmann equation, and the conserved density is a function of both space and velocity degrees of freedom. Interpretation of the Lynden-Bell equilibrium is problematic in stellar dynamics, since in contrast to two dimensions, in three dimensions equilibria do not exist under physical boundary conditions.

The origin of this work was the suggestion by Cross[18] that Marcus' dynamical simulations[19] of Jupiter's Red Spot might be explained in the terms of statistical mechanics. We hope in the future to address the application of these methods to the spot.

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[12]That is, up to symmetry breaking.

[13]Equilibrium statistical mechanics requires a Liouville theorem; the viewpoint of Ref. 10 furnishes a simple way to see that one exists. The fluid flow is given by geodesics on the group of volume-preserving diffeomorphisms on Ω. The Liouville theorem is then equivalent to the group property: The action of an element of the group preserves volumes in phase space. For an alternative and more direct argument, see T. D. Lee, Q. Appl. Math. **10**, 69 (1952).

[14]Euler flow also preserves connectivity (see Ref. 10). This property is shared by phase-space flows, which can nonetheless be mixing. So connectivity is not in itself an obstacle to ergodicity.

[15]J. Miller and P. B. Weichman (to be published).

[16]We would like to emphasize the distinction between point vortices and a continuous vorticity field: Whereas both the point vortex gas and a continuous vorticity field are sources for an incompressible *velocity* field, *the point vortex gas is itself compressible*, whereas a continuous vorticity field is *incompressible*.

[17]As kindly pointed out to us by E. Ott and D. Montgomery. See D. Lynden-Bell, Mon. Not. Roy. Astron. Soc. **136**, 101 (1967); S. Tremaine, M. Henon, and D. Lynden-Bell, Mon. Not. Roy. Astron. Soc. **219**, 285 (1986); J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton Univ. Press, Princeton, 1987); W. Saslaw, *Gravitational Physics of Stellar and Galactic Systems* (Cambridge Univ. Press, Cambridge, 1985).

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