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Determination of Correlation Spectra in Chaotic Systems

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We propose a new method for evaluating the decay rates of time correlations in chaotic dynamical systems, based on averaging over periodic orbits. We use a cycle expansion of a Fredholm determinant which is in practice superior to the corresponding expansion for the Ruelle ζ function. The method is tested in one-dimensional expanding maps with the resulting decay rates compared to those obtained in two other independent ways: by a perturbative calculation of the spectrum of the transfer operator and by direct numerical computations of time correlations. Moreover, we show that in general the decay rates are not simply related to generalized Lyapunov exponents.

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Time correlation functions play a central role in the study of dynamical systems.[1] The power spectra of these correlations are usually the only experimentally accessible quantities in chaotic and turbulent flows. On the other hand, the characterization of deterministic chaos in systems with few degrees of freedom has been achieved mainly by the calculation of Lyapunov exponents, fractal dimensions, and entropies.[2,3] At present, there exist for these quantities rather sophisticated numerical methods,[4,5] related to the dynamical ζ functions[6] by calculation of unstable periodic orbits. Nevertheless, these methods are not fitted for extracting the decay rates of time correlations, since ζ functions are meromorphic, i.e., have simple poles. In this Letter we present the first determination of these rates using periodic orbits. This has been done through the study of a Fredholm determinant d which in many cases (for example, axiom-A maps[6]) is an entire function in the complex plane. The decay rates of time correlations are determined to a high precision in terms of the zeros of d. Our technique is considerably more efficient than a direct analysis of the time signal by interpolating exponential methods.[7,8]

For simplicity, we focus our discussion on analytic expanding maps $f(x)$ of the interval which conserve the probability measure (with zero escape rate). For this case we can perform a perturbative calculation of the eigenvalues of the transfer Perron-Frobenius operator[6] which gives independent and very accurate results. However, most of our considerations apply to more general cases like hyperbolic repellors and higher-dimensional axiom-A (Ref. 6) attractors. For two observables A and B, the time correlation $C_{A,B}(\tau)$ is given by $\langle A(x_{n+\tau})B(x_n)\rangle - \langle A\rangle\langle B\rangle$, where the time average $\langle \cdots \rangle$ is assumed to converge and to define a unique ergodic probability measure (the natural measure[6]). It can be modulated and, in many situations, expressed as a sum of complex exponentials,

$$C_{A,B}(\tau) = \sum_{i=1}^{\infty} c_i e^{-\alpha_i \tau} e^{i\omega_i \tau}, \tag{1}$$

where the coefficients c_i depend on the choice of the observables but the exponents $\alpha_i - i\omega_i$, the resonances of the system, do not. Let us briefly sketch how these resonances, for axiom-A systems, can be related to the eigenvalues of the Perron-Frobenius operator $\mathcal{L}$, defined by the integral Fredholm equation

$$[\mathcal{L}\Phi](x) = \int \delta(x - f(y))\Phi(y)dy = \sum_{x=f(y)} \frac{1}{|D_y f|}\Phi(y), \tag{2}$$

where the sum extends over the preimages of x and D_y is the derivative taken at y. The integral kernel $G(x,y) = \delta(x - f(y))$ defines a compact operator having eigenvalues ν_i $(i = 0, 1, \ldots, \infty)$, which decrease exponentially with i for expanding maps.[9] For one-dimensional repellors, $-\ln|\nu_0|$ is the escape rate[10] from the invariant subset of f. If the measure is preserved by the time evolution, the first eigenvalue of $\mathcal{L}$ is $\nu_0=1$. Using the relation

$$\langle A(f^\tau(x))B(x)\rangle=\int dy\,A(y)[\mathcal{L}^\tau B\mu](y)\,,$$

one realizes that correlations decay as $C_{A,B}(\tau)=\sum_{i=1}^{\infty}c_i(\nu_i)^\tau$, where the real part of $-\ln(\nu_i)$ is the mixing rate α_i and the imaginary part is the oscillation frequency ω_i in (1). In order to determine the spectrum of $\mathcal{L}$, we study the Fredholm determinant $d(z)=\det[1-z\mathcal{L}]=\prod_k(1-z\nu_k)$ for complex z. Its zeros (repeated according to their multiplicity) give the reciprocal eigenvalues of $\mathcal{L}$. This is easily seen through a formal calculation, defined for z values inside the inverse spectral radius r of $\mathcal{L}$ ($||z\mathcal{L}||<1$), analytically continued outside this region. Using the Taylor expansion of $\ln(1-z\nu_k)$ we get

$$-\ln(d)=\sum_n\frac{z^n}{n}\sum_k\nu_k^n=\sum_n\frac{z^n}{n}\mathrm{Tr}\mathcal{L}^n\,. \tag{3}$$

The technical utility of introducing the Fredholm determinant stems from the possibility of calculating the trace in (3) directly from the integral kernel of $\mathcal{L}$, allowing us to connect d to the periodic cycles of f^n:

$$\mathrm{Tr}\mathcal{L}^n=\int dx\,\delta(x-f^n(x))=\sum_{\mathrm{fix}(f^n)}\frac{1}{|1-J|}\,, \tag{4}$$

where $J\equiv D_xf^n|_{x=x_{\mathrm{fix}}}$ and the sum extends over the fixed points x_{fix} of f^n. In higher-dimensional cases one replaces $|1-J|$ with $\det|\mathbf{1}-\mathbf{J}|$ in (4). Expanding the exponential function, we obtain the power series

$$d(z)=\prod_{n=1}^{\infty}\exp\left(-\frac{z^n}{n}\sum_{\mathrm{fix}(f^n)}\frac{1}{|1-J|}\right)$$
$$=1+c_1z+c_2z^2+\cdots\,, \tag{5}$$

where the coefficients c_m decrease exponentially[9,11] with m^2, implying that the ν_k decrease exponentially with k. Hence $d(z)$ is analytic in the whole complex plane. In our calculations, we numerically find the fixed points of f^n up to some given order n, expand the sum in (5) to this order, and use a zero-localizing routine to obtain the desired reciprocal eigenvalues. For expanding maps, $|J|>1$, so that $|1-J|^{-1}=|J|^{-1}\sum_{m=0}^{\infty}J^{-m}$ and the Fredholm determinant can be rewritten as a product over prime cycles (PC),[5]

$$d=\prod_m\prod_{p,x\in\mathrm{PC}(f^p)}\left(1-\frac{z^p}{|J|J^m}\right)\equiv\prod_m\zeta_m^{-1}\,.$$

In this way, d is given by an infinite product of inverse ζ functions defined as

$$\zeta_m^{-1}=\prod_{n=1}^{\infty}\exp\left(-\frac{z^n}{n}\sum_{\mathrm{fix}(f^n)}|J|^{-1}J^{-m}\right)\,. \tag{6}$$

It is worth stressing that all ζ_m^{-1}'s, in contrast to d, have poles between the zeros, as shown by Fig. 1. The zeros of ζ_0^{-1} are the inverse eigenvalues of $\mathcal{L}$,[9,11] but their numerical determination is difficult, due to the intrinsic pole structure. However, from the zeros of the polynomial (5), we obtain a large number of leading resonances to a high precision.

In order to illustrate our arguments, consider the Bernoulli map $f(x)=x/p$ for $0\le x\le p$ and $f(x)=(x-p)/q$ for $p<x\le 1$ ($q\equiv 1-p$), as well as the tent map where the latter branch is replaced by $f(x)=(1-x)/q$ for $p<x\le 1$. It is useful to consider a set of generalized Lyapunov exponents[3]

$$L(Q)=\lim_{n\to\infty}\frac{1}{n}\ln\langle|D_xf^n|^Q\rangle,\quad Q\in R\,. \tag{7}$$

They are related to a Ruelle zeta function $\bar\zeta_m$, defined in the same way as ζ_m in (6), but with J replaced by its absolute value $|J|$. The location of the first pole of $\bar\zeta_{m-1}(z)$ is at $|z|=\exp(-P_m)$, where P_m is called the topological pressure[6] of the weight function $|D_xf^n|^{-m}$, and $P_1=-\ln(\nu_0)$. For hyperbolic maps one has[12] $P_m=L(1-m)$. The generalized Lyapunov exponents for both the Bernoulli map and the tent map are $L(Q)=\ln(p^{1-Q}+q^{1-Q})$, since the averages (7) are given by a Bernoulli shift where $|D_xf|=p^{-1}$ with probability p and $|D_xf|=q^{-1}$ with probability q. On the other hand, an explicit calculation of the eigenvalues of the Perron-Frobenius operator gives[1] $\nu_k=p^{k+1}+q^{k+1}$ for the Bernoulli map and $\nu_k=p^{k+1}+(-1)^kq^{k+1}$ for the tent map. Note that $L(-m)=\ln(\nu_m)$ in piecewise-linear maps which conserve the probability measure and have positive Jacobians. It has thus been conjectured[13]

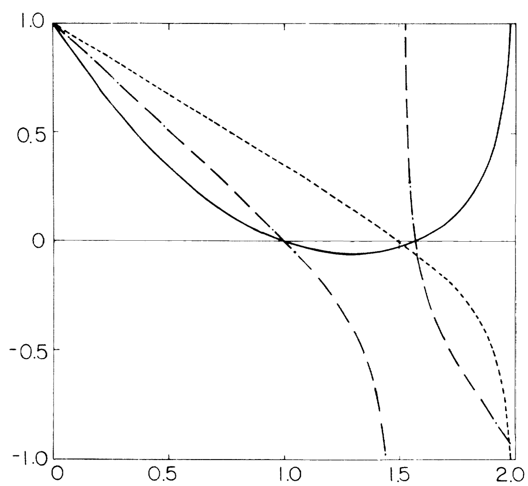

FIG. 1. Inverse dynamical zeta functions $\zeta_0^{-1}(z)$ and $\zeta_1^{-1}(z)$ (dot-dashed and dashed lines) with their product (solid line) vs z real, for a Bernoulli map ($p=0.8$) with the addition of a small nonlinear term. Observe the cancellation of the pole of ζ_0^{-1} in the product.

that the leading correlation decay is equal to the average inverse multiplier $\langle (D_x f)^{-1} \rangle$ in one-dimensional maps with an absolutely continuous measure [i.e., $\alpha_1 = L(-1)$, whenever J has fixed sign]. To show that such equalities are not generic, and that the first zero of ζ_m^{-1} $(m \geq 1)$ in general is different from the mth zero of the Fredholm determinant, we add a nonlinear term to the Bernoulli and the tent maps. In this case, we introduce a perturbative approach which allow us to obtain the eigenvalues of the Perron-Frobenius operator with very high precision and permits an independent test of the accuracy of our cycle expansion. In practice, we have added a parabolic function to the linear pieces of the inverse Bernoulli and tent maps, although all kind of smooth perturbations could be considered. The maps are, for $0 \leq x \leq p$,

$$f(x) = \{(h-p) + [(h-p)^2 + 4hx]^{1/2}\}/2h \,,$$

and, for $p < x \leq 1$,

$$f(x) = \{h_- + [h_-^2 + 4h(x-p)]^{1/2}\}/2h \,, \qquad (8a)$$

$$f(x) = \{h_+ - [h_+^2 - 4h(1-x)]^{1/2}\}/2h \,, \qquad (8b)$$

with $h_\pm = h \pm q$ and $q \equiv 1-p$. These maps reduce respectively to the Bernoulli map and to the tent map for $h \to 0$. In order to carry out the perturbative calculations, we consider a convenient basis in the functional space which $\mathcal{L}$ acts on, namely, the set of functions $\mathbf{u} = (u_0, u_1, \ldots)$ with $u_i = x^i$. Acting with $\mathcal{L}$ on $\mathbf{u}$, we get a matrix representation L of the Perron-Frobenius operator, i.e., $\mathcal{L} u_n = \sum_m u_m (\mathsf{L})_{mn}$ and from its diagonalization we obtain the eigenvalues $\{\nu_k\}$. For a piecewise-linear map, L is upper triangular and the eigenvalues are given by its diagonal elements. Nonzero elements $O(h)$ are introduced below the diagonal by a nonlinear perturbation of the map. In this case we find the eigenvalues of L by standard numerical methods. The order of the perturbation is determined by the size considered for L. In Fig. 2, a direct numerical computation of the autocorrelation of the observable $\sin(\pi x)$ is compared to the perturbative calculation for the map (8b), using a 100×100 matrix. On the other hand, via prime-cycle expansion,[5] we have obtained the various first poles of the ζ_m's as well as the

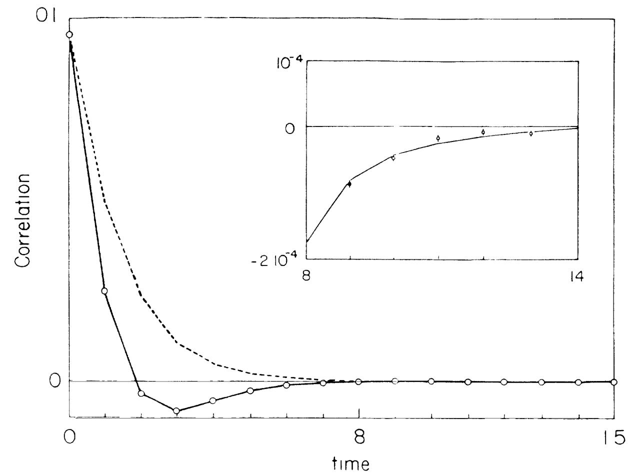

FIG. 2. The autocorrelation $C(\tau) = \langle \sin(\pi x_{n+\tau}) \sin(\pi x_n) \rangle - \langle \sin(\pi x) \rangle^2$ vs τ. The solid line is the perturbative result for map (8b) with $p = 0.8$ and $h = 0.1$; the dashed line is the unperturbed autocorrelation given by the corresponding tent map. The circles are obtained by a direct numerical iteration of (8b) for 10^8 steps. Inset: Magnification of the correlation tail with the error bars of the data.

TABLE I. For the map (8a) with $p = 0.8$ and $h = 0.1$, we give fourteen eigenvalues ν_m of $\mathcal{L}$ obtained by the perturbation theory and the corresponding inverses of the zeros of $d(z)$ obtained by the expansion (5), using quadruple precision and cycles of period ≤ 21. The number of significative digits of the zeros of $d(z)$ is given in the fourth column. The last column gives the first zero of $\zeta_m^{-1}(z)$.

m	Perturbation theory	Zeros of $d(z)$	Significant digits	$\exp[L(-m)]$
0	1.000 000 0	1.000 000 0	30	1.000 000 0
1	0.609 577 2	0.609 577 2	25	0.569 617 8
2	0.396 100 4	0.396 100 4	22	0.360 264 3
3	0.211 219 8 + i0.062 325 3	0.211 219 8 + i0.062 325 3	18	0.243 632 5
4	0.211 219 8 − i0.062 325 3	0.211 219 8 − i0.062 325 3	18	0.168 831 0
5	0.101 624 5 + i0.065 689 8	0.101 624 5 + i0.065 689 8	15	0.117 820 1
6	0.101 624 5 − i0.065 689 8	0.101 624 5 − i0.065 689 8	15	0.082 393 6
7	0.044 757 6 + i0.046 454 8	0.044 757 6 + i0.046 454 8	11	0.057 657 1
8	0.044 757 6 − i0.046 454 8	0.044 757 6 − i0.046 454 8	11	0.040 355 7
9	0.017 476 6 + i0.029 124 9	0.017 476 6 + i0.029 124 9	8	0.028 248 0
10	0.017 476 6 − i0.029 124 9	0.017 476 6 − i0.029 124 9	8	
11	0.005 562 3 + i0.016 978 5	0.005 560 3 + i0.016 979 1	5	
12	0.005 562 3 − i0.016 978 5	0.005 560 3 − i0.016 979 1	5	
13	0.000 896 9 + i0.009 358 5	0.000 792 4 + i0.009 228 3	3	
14	0.000 896 9 − i0.009 358 5	0.000 792 4 − i0.009 228 3	3	

TABLE II. Convergence of the cycle expansion for the initial three zeros of $d(z)$ as function of the cycle length.

Cycle length	First zero	Second zero	Third zero
10	0.999 999 978 057	0.609 582 926 685	0.395 815 287 444
11	1.000 000 000 289	0.609 577 043 174	0.396 109 891 634
12	0.999 999 999 997	0.609 577 170 192	0.396 100 162 372
13	1.000 000 000 000	0.609 577 167 745	0.396 100 450 887
14		0.609 577 167 759	0.396 100 448 279
15			0.396 100 448 392
16			0.396 100 448 394

zeros of the Fredholm determinant. Table I shows that the generalized Lyapunov exponents $L(Q=-m)$ are different from the eigenvalues ν_m, for a generic one-dimensional map. It also gives the accuracy of our cycle expansion method by the comparison between zeros of d and the independent perturbative estimate of the eigenvalues ν_m. In fact, our perturbative approach gives the resonances with the computer precision but only for analytic maps with sufficiently weak nonlinear part. In contrast, the Fredholm determinant provides us with resonances for any expanding analytic map. The fast convergence of the polynomial approximation of $d(z)$ with the order of the cycle length is shown in Table II. This method is superior to the cycle expansion of the ζ function[6] as well as to a direct analysis[7] of the time signal. Our scheme can also be applied in higher dimension as long as the symbolic dynamics has a finite grammar and the cycles of the system are bounded away from marginal stability. A detailed understanding of the theory of transfer operators and ζ functions is not required for the computation of the polynomial form (5). This makes our method an easy numerical tool. However, it is not suited for maps too close to the intermittent transition, where a probabilistic approach exists.[8] Moreover, the application to systems with infinite grammars, such as the Hénon map, is not straightforward, because of the slow convergence of the cycle expansion. We believe that this is still a major open problem in the analysis of dynamical systems.

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