Large-Order Behavior of the Perturbation Series for Superconductors near H_{c2}

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The perturbation series for superconductors near H_{c2} are studied in the two- and three-dimensional Ginzburg-Landau model. The large-order behavior is discussed first on a theoretical basis by an instanton method of Lipatov type. The results are compared with an eleventh-order calculation in 2D and a sixth-order one in 3D and show good agreement with the theoretical prediction. The conjectures based upon an Abrikosov lattice configuration seem to be ruled out.

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The discovery of the high-temperature superconductors has renewed interest in fluctuations in quasi-two-dimensional superconductors. Type-II superconductors have an Abrikosov vortex structure under a magnetic field and the vortex structure has been observed also for a high-temperature superconductor in a sufficient low-temperature region. However, near the upper critical field H_{c2}, the observation of Abrikosov vortex structure has not been reported, and a large fluctuation region appears near the transition temperature and H_{c2}.

In a strong magnetic field, the specific heat has a universal scaling form; the reduced temperature is scaled by a magnetic field. The superconducting fluctuation propagator in Ginzburg-Landau theory is quantized and the $n=0$ lowest Landau level provides the dominant contribution to the fluctuation in a strong magnetic field. This universal scaling function of the specific heat has been discussed by perturbation series[1-5] for the Ginzburg-Landau model in a magnetic field. The application to the high-temperature superconductor is of interest.[6]

The specific-heat perturbation series for the $n=0$ lowest Landau level is an asymptotic series. The low-temperature behavior (strong-coupling behavior) is obtained by the Padé or the Borel-Padé extrapolation method for this perturbation series. In an asymptotic perturbation series, the large-order behavior is governed by the existence of a tunneling or instanton solution of the classical equation of motion for a negative coupling constant. In type-II superconductors near H_{c2}, the Abrikosov triangle-vortex solution has been considered to be the classical solution relevant to the large-order behavior, since the lowest-energy configuration is a triangle-vortex lattice with a mean-field approximation. Ruggeri and Thouless[3] have conjectured for this reason that the large-order behavior is related to the Abrikosov ratio $\beta_A=\langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2$, which is evaluated as 1.16. We will show that our results do not support this conjecture.

We consider in this paper the large-order behavior of this perturbation series by more extended calculations and compare them to a new theoretical analysis of the instanton solutions. The free energy of the Ginzburg-Landau model is given by

$$F(\psi)=(1/2m)|(-i\hbar\nabla-2e\mathbf{A})\psi|^2+\alpha|\psi|^2+\tfrac{1}{2}\beta|\psi|^4. \quad (1)$$

We use the $\hbar=k_B=c=1$ units. By the choice of $\mathbf{A}=(0,xH,0)$ the order parameter $\psi(\mathbf{r})$ is expressed by

$$\psi(\mathbf{r})=\sum_q\sum_k a_{q,k}(L_yL_z)^{-1/2}(2eH/\pi)^{1/4}\exp(iqy+ikz)\exp\left[-eH\left(x-\frac{q}{2eH}\right)^2\right], \quad (2)$$

where $a_{q,k}$ is a complex field, and L_y,L_z are the lengths of the system. The free energy of (1) is written by

$$\int F(\psi)d^3r=\sum_q\sum_k|a_{q,k}|^2\left(\alpha_H+\frac{k^2}{2m}\right)+\sum_{q_1\cdots q_4}\sum_{k_1\cdots k_4}\frac{1}{2}\beta(L_yL_z)^{-1}\left(\frac{eH}{\pi}\right)^{1/2}\exp\left\{-\frac{1}{4eH}\left[\sum q_i^2-\frac{1}{4}\left(\sum q_i\right)^2\right]\right\}$$
$$\times\delta_{k_1+k_2,k_3+k_4}\delta_{q_1+q_2,q_3+q_4}a^*_{q_1k_1}a^*_{q_2k_2}a_{q_3k_3}a_{q_4k_4}. \quad (3)$$

The problem reduces to the functional integral about the complex variable $a_{q,k}$.

The Abrikosov solution is easily obtained by the assumption that a_q in (2) has a δ-function form, $a_q=\sum_n\delta(q-\lambda n)$, and that the solution has a periodic lattice structure. The Abrikosov solution is given by $C_n=C_{n+2}$,[7,8]

$$\psi(\mathbf{r})=\sum_{n=-\infty}^{\infty}C_ne^{in\lambda y}e^{-(1/2)(x-\lambda n)^2}, \quad (4)$$

where $\lambda=3^{1/4}\sqrt{\pi}$ for the triangular-vortex lattice. From this equation (4), the Abrikosov ratio β_A is evaluated as

1.16.

In a two-dimensional perturbation about β, the Gaussian integral about $a_{q,k}$ (k is set to zero) reduces to a counting problem. The two-dimensional free energy f_{2D} is expressed in the following perturbation series:[3]

$$G_{2D}=\frac{T}{L_z}\frac{eH}{\pi}\left[\ln\left(\frac{\tilde{\alpha}}{\pi T}\right)+f_{2D}(x)\right], \tag{5}$$

$$f_{2D}(x)=-\sum_{\text{graph}}(-4x)^n\frac{1}{\tilde{T}G}\sim\sum C_n x^n, \tag{6}$$

where $\tilde{T}$ is a number of the Euler paths and G is a symmetry factor. The scaling parameter x is defined by $\beta eHT/2\pi L_z\tilde{\alpha}^2$. We use a new reduced relative temperature $\tilde{\alpha}$ instead of α_H, since the renormalization of the Hartree term is included, $\alpha_H=\alpha+(e/2m)H=\tilde{\alpha}(1-4x)^3$. The definition of x manifests the scaling $\tilde{\alpha}$ by a magnetic field H. For example, the transition region of the specific heat is scaled by $\sqrt{H}$. The perturbation series of $f_{2D}(x)$ is an asymptotic expansion. The large-order behavior of this series is important for the study of the low-temperature region.

We now derive the asymptotic behavior theoretically. The free energy of (3) in two dimensions becomes, by the use of the rescaled quantities, $b_q=[(2\pi T/\alpha_H L_y)\times\gamma^{1/2}]^{-1/2}a_q$, $\gamma=1/4eH$, q is scaled by $1/\sqrt{\gamma}$:

$$A=\int dq|b_q|^2+\frac{x}{\sqrt{\pi}}\int dq_1dq_2dq_3dq_4b^*_{q_1}b^*_{q_2}b_{q_3}b_{q_4}\exp\left\{-\left[\sum q_i^2-\frac{1}{4}\left(\sum q_i\right)^2\right]\right\}\delta_{q_1+q_2,q_3+q_4}. \tag{7}$$

We shall follow Lipatov's method[9,10] and determine the large-order behavior of the expansion in powers of x from the nature of the tunneling (or instanton) solution to the classical equations of motion. These equations have a class of solutions of Gaussian type,[7-11]

$$b_q=Ce^{-\sigma q^2}, \tag{8}$$

where C and σ are parameters which will be determined later. By further rescaling, $C=D(2\sigma/\pi)^{1/4}$, the action A of Eq. (7) becomes, after integration over the q_i's,

$$A(D,\sigma)=|D|^2+[x\sqrt{\sigma}/(1+\sigma)]|D|^4. \tag{9}$$

The definition of x differs from (6) using α_H instead of $\tilde{\alpha}$ which is conveniently used for the perturbation series. However, the large-order behavior is not affected by this difference, and we use the same notation as x. Up to now, D and σ are free parameters; the leading large order is related to the maximum value of the coefficient of x with respect to σ. The coefficient $\sqrt{\sigma}/(1+\sigma)$ takes a maximum value of $\frac{1}{2}$ when σ equals unity. Then the remaining procedure for the calculation of the large order reduces to the usual derivation.[8-10] We consider the following quantity I, which generates the nth order in perturbation theory,

$$I=\int dD\oint\frac{dx}{x}\exp(-n\ln x-D^2-\tfrac{1}{2}xD^4). \tag{10}$$

The integration over x runs around the cut on the real negative x axis, and we obtain the large-n behavior from the saddle-point equations for x and D:

$$n/x+D^4/2=0, \tag{11}$$

$$D+xD^3=0. \tag{12}$$

The solutions $x=-1/2n$, $D^2=2n$ give the large-order behavior

$$I\sim(n/e)^n(-1)^n2^n. \tag{13}$$

Using Stirling's approximation formula, the coefficient of $(-1)^n n!$ is 2^n instead of the Ruggeri-Thouless–conjectured value $(2\times1.16)^n$. This difference was to be expected since we are looking at the singularity of the free energy at the region in the complex H plane, whereas the transition from a homogeneous to an Abrikosov lattice, which was used by Ruggeri and Thouless, is a singularity at the finite value of x, which corresponds to H_{c2}.

In the three-dimensional case, we start again from the free energy (3) in the thermodynamic limit, rescale $q\to q\sqrt{4eH}$ and $k\to k\sqrt{2m\alpha_H}$, and obtain

$$A=\frac{1}{T}\int F(\psi)d^3r=\frac{\alpha_H L_y L_z}{T}\sqrt{8meH\alpha_H}\int dq\,dk|a_{q,k}|^2(1+k^2)$$
$$+\frac{\beta L_y^2L_z^2}{2T(2\pi)^6}\left(\frac{eH}{\pi}\right)^{1/2}(8meH\alpha_H)^{3/2}\int dq_1\cdots dq_4dk_1\cdots dk_4a^*_{q_1k_1}a^*_{q_2k_2}a_{q_3k_3}a_{q_4k_4}$$
$$\times\delta(q_1+q_2-q_3-q_4)\delta(k_1+k_2-k_3-k_4)$$
$$\times\exp\left\{-\left[\sum_{i=1}^4q_i^2-\frac{1}{4}\left(\sum_{i=1}^4q_i\right)^2\right]\right\}. \tag{14}$$

As before, we obtain a classical solution with a Gaussian dependence on q,

$$a_{q,k}=b_ke^{-\sigma q^2}, \tag{15}$$

and this gives an action

$$A\{b\} = \left(\frac{\pi}{2\sigma}\right)^{1/2} \frac{\alpha_H L_y L_z}{(2\pi)^2 T} \sqrt{8meH\alpha_H} \int dk\, |b_k|^2 (1+k^2)$$
$$+ \frac{\pi^{3/2}\beta L_y^2 L_z^2}{4T(2\pi)^6 \sqrt{\sigma}(1+\sigma)} \left(\frac{eH}{\pi}\right)^{1/2} (8meH\alpha_H)^{3/2} \int dk_1 \cdots dk_4\, \delta(k_1+k_2-k_3-k_4)$$
$$\times \exp\left\{ -\left[\sum_{i=1}^{4} q_i^2 - \frac{1}{4}\left(\sum_{i=1}^{4} q_i\right)^2 \right] \right\} b_{k_1}^* b_{k_2}^* b_{k_3} b_{k_4} . \qquad (16)$$

We go now to real space,

$$b(t) = \int b_k e^{ikt} dk , \qquad (17)$$

assuming that $b(t)$ is real, and rescale $b(t)$

$$b(t) \to \left[\frac{(2\pi)^3 T (2\sigma)^{1/2}}{4 L_y L_z \alpha_H^{3/2} (2m)^{1/2} (eH)^{1/2} \pi^{1/2}} \right]^{1/2} b(t) , \qquad (18)$$

and obtain

$$A(b) = \tfrac{1}{2} \int dt [b^2(t) + \dot{b}^2(t)] + \tfrac{1}{2} x \int dt\, b^4(t) . \qquad (19)$$

One recovers here the (imaginary-time) action for a one-dimensional anharmonic oscillator with Hamiltonian

$$H = \tfrac{1}{2}(p^2+q^2) + \tfrac{1}{2} x q^4 . \qquad (20)$$

In three dimensions, we use a different definition for a scaling parameter x, $x = \beta eHT\sqrt{2m}/8\pi\alpha_H^{3/2}$. The difference is due to the appearance of the k integral in three dimensions. This also manifests the scaling behavior between α_H and H. The large-order behavior of the perturbation series has been studied long ago for this problem,[8-12] and the result is that the coefficient of x^n behaves for large n as $(-1)^n n! (\frac{3}{2})^n$. This completes the derivation of the large-order behavior in three dimensions.

We have extended the calculation by Ruggeri and Thouless up to x^{11} which was done before up to x^6 in the two-dimensional case,

$$f_{2D}(x) = -2x - x^2 + \frac{38}{9} x^3 - \frac{1199}{30} x^4 + 471.396\,594\,517 x^5$$
$$- 6471.562\,574\,96 x^6 + 101\,279.327\,846 x^7 - 1\,779\,798.787\,59 x^8$$
$$+ 34\,709\,019.6144 x^9 - 744\,093\,435.668 x^{10} + 10\,373\,276\,492.7 x^{11} . \qquad (21)$$

The calculation was done with a computer program which generates all relevant graphs by a tree-sorting method. The symmetry factor G in (6) is obtained automatically by counting all the combinations. The number of Euler paths $\tilde{T}$ is evaluated by the determinant of the adjacency matrix which represents the corresponding graph. The ratio C_n/nC_{n-1} is given in Table I and is also plotted in Fig. 1 as a function of $1/n$. If C_n has an asymptotic form, $C_n \simeq n! \times a^n n^b c$ for large n, this ratio becomes $a[n/(n-1)]^b \simeq a[1+b/(n-1)]$, the value at $1/n = 0$ gives the value of a, and b is estimated from the slope of the line.

Contrary to Ruggeri and Thouless's conjecture, the value of a deviates significantly from the conjectured value 2×1.16 which is indicated by an arrow in Fig. 1. In Fig. 1, the extrapolation line is almost linear. The end point of the estimated line deviates slightly from 2, but we consider that this is caused by a linear extrapolation. Therefore, we see the agreement between our theoretical estimate (13) and the eleventh-order perturbation about the value of a as 2.

TABLE I. The ratios of the coefficients in Eq. (21).

n	C_n/nC_{n-1}
11	2.125 766
10	2.143 804
9	2.166 850
8	2.196 646
7	2.235 700
6	2.288 081

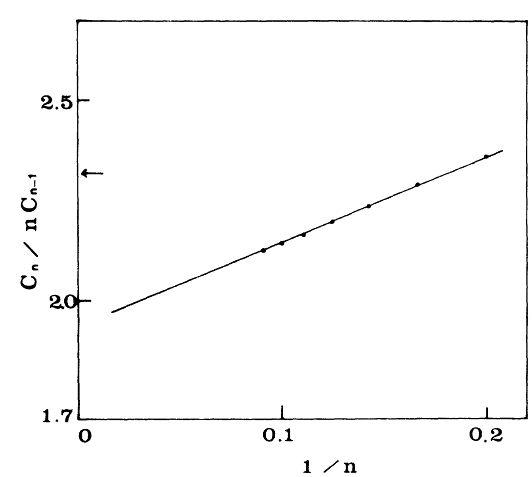

FIG. 1. C_n/nC_{n-1} vs $1/n$. The ratios in Table I are plotted. The straight line is a guideline. The arrow shows the predicted value of Ruggeri and Thouless, 2×1.16.

The lower bound for the value of C_n is obtained as

$$n!2^n\sqrt{n}(2e^{-7/2}/\sqrt{\pi}) < |C_n| . \qquad (22)$$

This lower bound becomes $n!2^n n^{1/2}\times 0.03406$. If we assume that $a=2$ and $b=\frac{1}{2}$, the coefficient c of the asymptotic behavior is estimated from (21), $c \simeq C_n/n!2^n n^{1/2}$. The estimation for c becomes $c=0.06096(n=8)$, $0.06227(n=9)$, $0.06332(n=10)$, $0.06417(n=11)$. Therefore, our numerically estimated value is indeed larger than the lower bound (22). For the three-dimensional case, we compare our theoretical estimate with the perturbation series up to sixth order, evaluated by Ruggeri and Thouless. From the ratios C_n/nC_{n-1} for $n=5$ and 6, we find that our estimated value $a=\frac{3}{2}$ is also consistent with the explicit calculation.

From our results, we conclude that the Abrikosov triangular-lattice solution cannot be detected in large perturbation order. As we have shown, the Abrikosov solution is not related to the perturbation series expressed by Eq. (22). Our discussion is based upon the Ginzburg-Landau model, which does not take account of vortex-vortex interactions. This vortex-vortex interaction may be important for the existence of the vortex lattice structure. What we have shown in this paper is that there is a region near H_{c2} where the fluctuation is large and that the Abrikosov vortex lattice is not stable. This may be related to the apparent unobservability of the vortex lattice in high-T_c superconductors near H_{c2}.

It is interesting to note that the similar large-order behavior appears in the perturbational calculation of the density of states in the lowest Landau level.[13,14] The same Euler-number factor appears in the perturbation. In this case, the large-order behavior can be obtained by the exact recurrence equation.[14] In this density-of-states problem, the parameter a is also an integer in large perturbation order.

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