Scaling Law for Dynamical Hysteresis

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A one-dimensional theory of dynamical hysteresis in periodically switched bistable systems is formulated. The predictions of the theory are tested and verified by experiments on a bistable semiconductor laser. We find that the increase of the area of the hysteresis loop scales as the two-thirds power of the switching frequency. From this, we deduce a law for the increase in input power necessary to maintain repetitive switching of the bistable system as the driving frequency is increased.

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Bistable systems used as switches are a fundamental component of electronic and optical devices.[1] The conceptual understanding of switching dynamics in bistable systems is thus of great scientific and technological interest. At first glance, bistable devices in different applications like optical and electronic switches appear rather different. However, as shown in recent studies by Mitschke *et al.*[2] and Boden, Mitschke, and Mandel,[3] simple models can be used to analyze a large variety of bistable systems provided they are described effectively by a single dynamical equation. The goal of this paper is to reveal a generic scaling property of the switching dynamics. We will explore the relation between switching frequency and power input necessary for repetitive switching and illustrate the theory with measurements on a bistable laser diode. Our analysis is based on the theory of dynamical bifurcations.[4]

The steady-state input-output characteristic (dashed curve) of a typical bistable system is shown in Fig. 1. The value of a control parameter F determines the output variable, depending on the history of the system. The output follows the lower branch as F is increased adiabatically from very small values; at the limit point I_1 the output jumps to the upper branch. On the return path, the output remains on the upper branch and jumps down to the lower branch at point I_2. A hysteresis loop is described in this process. If the control parameter is periodically modulated, instead of being adiabatically changed, the system displays a delay in making the transition to the upper state. A delayed transition begins at the value F_{del} of the control parameter, which depends on the modulation frequency Ω and the amplitude of modulation A. The hysteresis loop area increases with frequency. We address the following questions in this paper: How do F_{del} and the loop area change with frequency and modulation amplitude? If we consider a bistable element that has to be repetitively switched on and off (such as a memory element in an optical, electronic, or magnetic system), how does the amplitude of modulation necessary to produce reliable switching increase with frequency? We examine these fundamental issues analytically and test our conclusions on an optical bistable system.

We use a generic one-dimensional model for a switched bistable system; the overdamped dynamics of a particle in a quartic double-well potential with a periodic driving force, i.e.,

$$\frac{dx}{dt} = ax - bx^3 + F(t) \,, \qquad (1)$$

where the dynamic control parameter F is given by $F(t) = A\sin(\Omega t)$. Yamada has provided a detailed model to justify the analysis of longitudinal-mode bistability in a semiconductor laser in terms of a quartic potential such as that underlying the motion described by Eq. (1).[5] Changing the control parameter adiabatically, the stationary solution of (1) follows the curve $f(x) = bx^3 - ax$ on the stable branches up to the limit points I_1, I_2

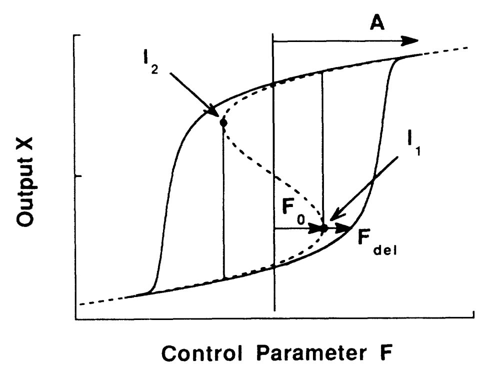

FIG. 1. The steady-state or adiabatic hysteresis loop for a bistable system is shown together with one obtained for periodic modulation of the control parameter. The quantities F_0, F_{del}, and A defined in the text are illustrated. The S curve (dashed line) associated with the bistability is shown for reference. Also indicated are the inflexion points I_1 and I_2.

given by

$$I_1=\left\{x=x_0\equiv-\left(\frac{a}{3b}\right)^{1/2},\ F=F_0\equiv\left(\frac{4a^3}{27b}\right)^{1/2}\right\},$$
$$I_2=(x=-x_0,\ F=-F_0)\,. \tag{2}$$

When the system reaches a limit point, the trajectory jumps to the adjacent stable branch (see Fig. 1). Increasing the driving frequency Ω, the hysteresis loop becomes larger since the limit point is delayed.[4(b)] If we expand Eq. (1) about the limit point I_1 up to second-order terms in the deviation y, with $x=-\sqrt{a/3b}+y$ and let $F(t)=F_0+\tilde{A}\sin(\Omega t)$, the resulting Ricatti-type differential equation

$$\frac{dy}{dt}=\sqrt{3ab}\,y^2+\tilde{A}\sin(\Omega t) \tag{3}$$

is characteristic of the dynamical behavior of a swept limit point. In this local description, the modulation amplitude $\tilde{A}$ is related to the parameters of the original problem by $\tilde{A}^2=A^2-F_0^2$ in the low modulation frequency limit. A transformation of variables $y(t)=[du(t)/dt]/\sqrt{3ab}\,u(t)$ is used to obtain a Mathieu equation. In the limit $\Omega\to 0$ [i.e., $\sin(\Omega t)\approx\Omega t$], it reduces to

$$\frac{d^2u(s)}{ds^2}+s\lambda^2u(s)=0\,, \tag{4}$$

where $\lambda^2=1/3ab\,\Omega^2\tilde{A}^2$ and $s=A\sqrt{3ab}\,\Omega t$. Equation (4) may be solved in terms of Airy functions, i.e., $u(s)=\mathrm{Ai}(-\lambda^{2/3}s)$. The solution in the original variable $y(t)$ is then given as

$$y(t)=\lambda^{2/3}\tilde{A}\,\Omega\,\frac{\mathrm{Ai}'(-\lambda^{2/3}\sqrt{3ab}\,\tilde{A}\,\Omega t)}{\mathrm{Ai}(-\lambda^{2/3}\sqrt{3ab}\,\tilde{A}\,\Omega t)}\,. \tag{5}$$

$F_{\rm del}$ is the delayed value of the control parameter as shown in Fig. 1 and is calculated from the zero of the derivative of the Airy function.[6] Our treatment is similar to that of Mandel.[4(a)] The delayed critical value of the control parameter is given for frequencies small compared to the curvature of one of the unperturbed potential wells (the well frequency) by

$$F_{\rm del}=F_0+K_1[\Omega^2(A^2-F_0^2)]^{1/3}\,, \tag{6}$$

where K_1 is determined in terms of a,b and the known properties of the Airy function.

Equation (6) reveals that the value of the control parameter at the switching point is delayed from the static value by an amount proportional to the $\frac{2}{3}$ power of the frequency of modulation. The dependence of $F_{\rm del}$ on the amplitude of modulation is also contained in Eq. (6). We note that A is assumed to be appreciably larger than the minimum amplitude necessary for switching. The regime of validity of these theoretical results has been verified through numerical computation [Fig. 2(a)], which shows the regime of validity of Eq. (6) to be up to

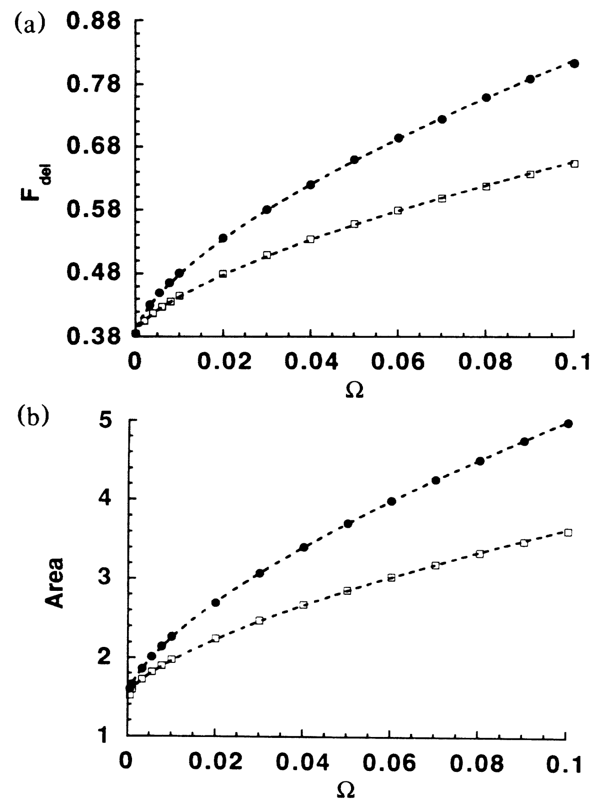

FIG. 2. Comparison of numerical computation and analytic theory for two different values of modulation amplitude A. F_0 is 0.3849 $[(\frac{4}{27})^{1/2}]$. Both $F_{\rm del}$ and the hysteresis loop area are found to scale as the two-thirds power of the modulation frequency. Values for $A=4$ (solid circles) and values for $A=2$ (open squares) are shown; the dashed lines are one-parameter least-squares fits by Eq. (6) for $F_{\rm del}$ and Eq. (7) for the area. (a) $F_{\rm del}$ vs frequency. From the fits, $K_1=0.803$ for $A=4$ and $K_1=0.798$ for $A=2$. (b) Loop area vs frequency. From the fits, $K_2=6.47$ for $A=4$ and $K_2=6.35$ for $A=2$.

frequencies comparable to the well frequency. The values of a and b are taken to be unity in the computations. In Fig. 2(a) we show the numerically calculated value of $F_{\rm del}$ versus frequency for two different values of modulation amplitude A. Also shown are least-squares fits to the data using Eq. (6) with only one free parameter, K_1. Note that the values obtained for K_1 from the fits (see the figure caption) are essentially constant for both values of modulation amplitude A. The hysteresis loop area is given by

$$\mathrm{Area}(A,\Omega)=\mathrm{Area}(A,0)+K_2[\Omega^2(A^2-F_0^2)]^{1/3} \tag{7}$$

and compared to numerical computation in Fig. 2(b). Since K_2 is related to the zero of the Airy function appearing in Eq. (5), we chose to treat it as the single adjustable fitting parameter when comparing the theory to the experimental data. Note that K_2 was found to be constant to a very good approximation, as predicted by the theory.

From these results we deduce that the amplitude of the periodic modulation necessary for reliable repetitive switching of a bistable device will have to be increased with frequency due to the increase in the control parameter value at which the switching point occurs. In order to have reliable switching, the amplitude of the modulation must exceed the critical value F_{del} by a certain amount, i.e., $A_s = \alpha F_{del}$, where α is larger than 1. This yields self-consistently, together with Eq. (6), the equation

$$A_s = \alpha F_0 + \alpha K_1 [\Omega^2 (A_s^2 - F_0^2)]^{1/3}. \quad (8)$$

The solution of this equation gives the value of the modulation amplitude A_s for reliable repetitive switching. A_s also increases with a $\frac{2}{3}$ power law with respect to the frequency,

$$A_s(\Omega) \approx A_s(0) + \alpha K_1 \{\Omega^2 [A_s^2(0) - F_0^2]\}^{1/3}. \quad (9)$$

The input power from the modulating device is proportional to the square of the amplitude of modulation. Thus, in this approximation, the input power necessary for repetitive switching increases in leading order with a $\frac{2}{3}$ power law in the frequency.

A test of these theoretical predictions is provided by experimental measurements of bistable switching of longitudinal modes in a semiconductor laser diode. Deterministic bistable switching has been observed in the intensity of a longitudinal mode of a two-mode laser diode.[7] The role of control parameter is taken by the injection current, while the intensity of one of the two longitudinal modes is the output variable. As the injection currents is swept over a certain range, this intensity makes a transition from a very small "OFF" value to a high intensity "ON" state. A periodic modulation of the injection current results in a hysteresis loop when the output intensity is plotted versus the injection current. A theoretical discussion of hysteresis in modulated, bistable, two-mode lasers was given by Agarwal and Shenoy.[8]

In our experiments, a transverse-junction-stripe semiconductor laser was temperature stabilized to a precision of better than 0.01 K. A careful search of temperature-current parameter space identified several operating points at which deterministic bistable switching from one longitudinal mode to another could be observed.[9] The laser was mounted inside an enclosure purged with dry nitrogen. The entire system is vibration isolated and a Faraday isolator is used to prevent feedback due to external reflections. The light from the laser is collimated and expanded with a telescope and is then incident on a diffraction grating that separates the intensities of the two longitudinal modes. A fast photodiode detects the light intensity of a single longitudinal mode and its output is digitized and stored by a digital oscilloscope. The injection current is modulated and the time variation in the intensity of the mode is recorded and converted to a hysteresis loop on a microcomputer. An average over several thousand loops is performed in order to average over the effects of noise-induced switching.

From the dynamical hysteresis loops recorded, we can make a direct comparison of the experimental results with the predictions of the theory, Eqs. (6) and (7). Figure 3(a) shows the dependence on modulation frequency of $F_{del} - F_0$. The measurements were carried out for two different amplitudes of modulation. The dashed lines are fits to the data with Eq. (6), where the parameters F_0 and A are experimentally determined while K_1 is determined by a least-squares fit. The parameter K_1 is essentially unchanged in the two curves. Figure 3(b) shows the results of comparison of the Eq. (7) with experimental measurements of the loop area. Once again, the pa-

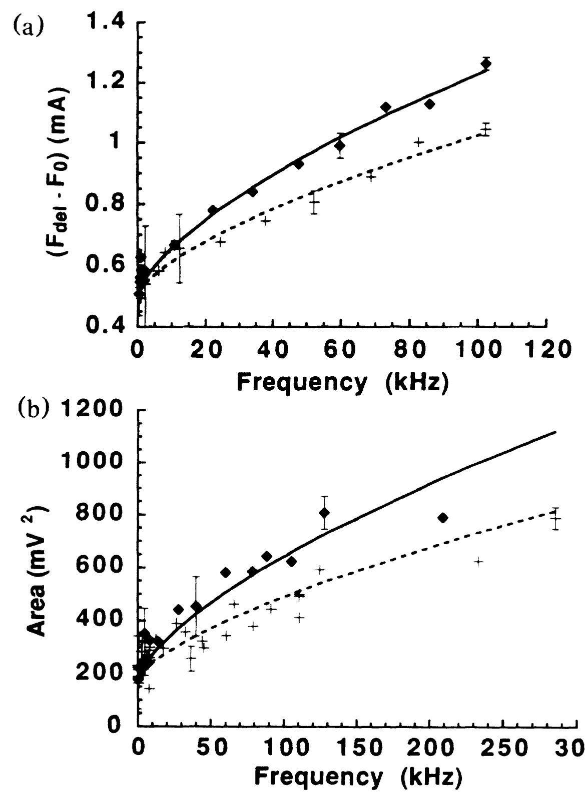

FIG. 3. Experimental measurements of (a) F_{del} and (b) hysteresis loop area for a bistable semiconductor laser diode. F_0 was measured to be 0.5 mA. (a) Delay vs frequency data for two values of modulation amplitude A. Data for $A = 3.98$ mA (solid diamonds) and a one-parameter fit by Eq. (6) (solid line) with $K_1 = 1.355 \times 10^{-4}$. Data for $A = 2.70$ mA (plus signs) and a fit by Eq. (6) (dashed line) with $K_1 = 1.270 \times 10^{-4}$. The units of K_1 are mA$^{1/3}$Hz$^{-2/3}$. (b) Area vs frequency for two modulation amplitudes. Data for $A = 4.4$ mA (solid diamonds) and a one-parameter fit by Eq. (7) (solid line) with $K_2 = 0.082$. Data for $A = 2.95$ mA (plus signs) and a fit by Eq. (7) (dashed line) with $K_2 = 0.073$. The units of K_2 are mV^2(mA Hz)$^{-2/3}$. Both F_{del} and the hysteresis loop area were found to scale as the two-thirds power of the modulation frequency. Error bars are shown that represent a standard deviation for the measurements of F_{del} and the loop area.

rameter K_2 is the only one determined by a least-squares fitting procedure.

The $\frac{2}{3}$ power law is seen to be very well confirmed in the range of frequencies studied. The agreement between theory and experiment demonstrates the validity of Eqs. (6) and (7), and therefore of the generic model (1), for the bistable operation of the two-mode semiconductor laser.

The model of bistability examined here is generic in one dimension, and the scaling laws it provides should be valid for a large class of bistable devices, optical or not. Indeed, simple models similar to Eq. (1) have been shown recently to provide scaling laws that fully agree with experimental results for widely different systems such as nonlinear interference filters,[10] optically thick molecular bistable systems,[11] electronic circuits,[2] and optically thin atomic bistable systems.[2] This clearly indicates that the scaling laws do not depend on the specific underlying physics of these systems, but are a manifestation of the nonlinear nature of their dynamics. However, the reduction to one dimension is crucial; for instance, in a recent paper, Rao, Krishnamurthy, and Pandit[12] show from numerical computations for a spin system subject to a periodic magnetic field that the hysteresis loop area scales with the two-thirds power of the field amplitude as in our Eq. (7) but that the exponent for frequency scaling is one-third, half that obtained here. This difference can be explained by the fact that their model is infinite dimensional and the static steady state has only a discontinuity: it corresponds to a hysteresis of vanishing width but finite amplitude. We note further that our results are restricted to the range of modulation frequencies that are small compared to the well frequency, which permits a one-dimensional description of the system dynamics. Of course, this is the regime of useful frequencies for a bistable switching device; at higher frequencies the shape of the hysteresis loop is eventually grossly distorted and clean ON-OFF switching is no longer observed. Further, delayed bifurcations are only one particular contribution which will increase the input power necessary for repetitive switching. In many devices this may be only a component of the increasing losses incurred in traversing a hysteresis loop at higher frequencies. However, it will always be a fundamental factor in considerations of the input power necessary to obtain repetitive switching in systems that are of the generic class defined by Eq. (1). We hope to stimulate experiments on other bistable systems to delineate the regime of validity of these results.

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