Extreme Quantum Entanglement in a Superposition of Macroscopically Distinct States

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(Received 29 May 1990)

A Bell inequality is derived for a state of n spin-$\frac{1}{2}$ particles which superposes two macroscopically distinct states. Quantum mechanics violates this inequality by an amount that grows exponentially with n.

PACS numbers: 03.65.Bz

Greenberger, Horne, and Zeilinger[1] (GHZ) have recently described a state $|\Phi\rangle$ of four spin-$\frac{1}{2}$ particles with the following remarkable property:

A group of spin-correlation experiments performed on four widely separated particles in the GHZ state reveals certain strong spin correlations to which the reasoning of Einstein, Podolsky, and Rosen[2] (EPR) can be directly applied. The EPR argument forces one to conclude that a new type of spin-correlation experiment in the GHZ state must *always* yield a certain outcome. One can, however, demonstrate by a simple quantum-mechanical calculation that in fact this outcome can *never* occur in the GHZ state.

This refutation of EPR is strikingly more direct than the one Bell's theorem[3] provides for Bohm's version[4] of EPR. In Bell's example the EPR argument gives a lower bound to the probability of a certain outcome to a new experiment ($33\frac{1}{3}$% in one simple case[5]), which, however, exceeds the probability quantum mechanics actually gives for that outcome (25%). In the GHZ experiment the corresponding "lower bound" is 100% and the result required by quantum mechanics, 0%. The refutation is not only stronger—it is no longer statistical and can be accomplished in a single run.

This all versus nothing demolition of EPR follows directly from a simple quantum-mechanical calculation of the data produced in the new experiment. The same point cannot, however, be inferred directly from the actual data collected in some nonideal laboratory realization of the GHZ experiment, because, for example, the less than perfect efficiency of real detectors weakens the observed spin correlations from the strong ideal form on which the EPR argument relies. Faced with this problem in the Bohm-EPR experiment, Clauser *et al.*[6] derived a correlation inequality whose validity is necessary for the observed correlations to be consistent with a very general probabilistic locality condition. The quantum theoretic predictions for the EPR correlations can exceed the bound imposed by this inequality by a factor as large as $\sqrt{2}$, allowing significant room for the softening effect of the imperfections in real experimental attempts to demonstrate quantum nonlocality.

Because the violation of the EPR locality condition is so much stronger in the GHZ state than in the two-particle spin-singlet state of the Bohm-EPR experiment, one might expect that correlation inequalities could be found for the GHZ experiment that were more strongly violated by the results predicted by the quantum theory. In this paper I show that this is indeed the case, deriving simple correlation inequalities for n-spin versions of the GHZ experiment. The data predicted by the quantum theory exceed these bounds by a factor that grows exponentially with n.

Because these experiments require the use of n independent detectors, whose joint efficiency necessarily declines exponentially in n, this does not establish that the n-particle GHZ states (which are also more and more difficult to prepare with increasing n) are necessarily a promising arena in which to eliminate such loopholes as remain in the direct experimental demonstration[7] of quantum nonlocality. They do, however, provide a striking theoretical demonstration of the surprising fact that there is no limit to the amount by which the correlations in a quantum-mechanically entangled state can exceed the limits imposed by a Bell inequality.

An n-particle state with these strange properties is simply

$$|\Phi\rangle=(1/\sqrt{2})(|\uparrow\uparrow\cdots\uparrow\rangle+i|\downarrow\downarrow\cdots\downarrow\rangle)\,, \tag{1}$$

where $\uparrow$ or $\downarrow$ in the jth position refers to the component of the jth particle along its own z axis. If one is thinking in terms of a real experiment, then it makes sense to regard the n particles as flying apart from some common source, taking the z axis for each particle to be along its direction of motion, and its x and y axes along any two orthogonal directions perpendicular to its line of flight. (The argument also works, however, if each particle is provided with its own arbitrarily oriented Cartesian triad.)

Everything of interest to us about the state $|\Phi\rangle$ follows from the easily confirmed fact that it is an eigenstate of the operator

$$A=\frac{1}{2i}\left[\prod_{j=1}^{n}(\sigma_x^j+i\sigma_y^j)-\prod_{j=1}^{n}(\sigma_x^j-i\sigma_y^j)\right], \tag{2}$$

with eigenvalue 2^{n-1}.

If one takes the diagonal matrix element of A in the state $|\Phi\rangle$ and expands the products, one finds

$$\begin{aligned}2^{n-1} = &\langle\Phi|\sigma_y^1\sigma_x^2\cdots\sigma_x^n|\Phi\rangle + \cdots\\ &-\langle\Phi|\sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\cdots\sigma_x^n|\Phi\rangle - \cdots\\ &+\langle\Phi|\sigma_y^1\cdots\sigma_y^5\sigma_x^6\cdots\sigma_x^n|\Phi\rangle + \cdots\\ &-\langle\Phi|\sigma_y^1\cdots\sigma_y^7\sigma_x^8\cdots\sigma_x^n|\Phi\rangle - \cdots\\ &+\cdots,\end{aligned} \tag{3}$$

where each line of (3) contains all distinct permutations of the subscripts that give distinct products.

The total number of terms in (3) is

$$\sum_{j\,\mathrm{odd}}\binom{n}{j} = 2^{n-1}. \tag{4}$$

Since each term must lie between -1 and 1, for the combination of correlation functions in (3) to add up to 2^{n-1}, each term must have its extremal value $+1$ or -1, and $|\Phi\rangle$ must therefore be an eigenstate of the operators appearing in every term. This observation leads directly to the EPR argument and its immediate refutation, as discussed elsewhere.[8] We do not pursue this line of thought here, because we are interested in the case where the measurements are imperfect, and the observed correlation functions $E_{\mu_1\cdots\mu_n}$ fail to attain the extreme values $\langle\Phi|\sigma_{\mu_1}^1\cdots\sigma_{\mu_n}^n|\Phi\rangle = \pm 1$ that they assume in the ideal case.

We therefore inquire whether the measured distribution functions $P_{\mu_1\cdots\mu_n}(m_1\cdots m_n)$ (with each μ_j either x or y and each m_j being $\uparrow$ or $\downarrow$) that describe the outcomes of the 2^{n-1} different kinds of experiments that must be performed on n particles in the state $|\Phi\rangle$ to yield the correlation functions appearing in (3) can all be represented in the conditionally independent form

$$P_{\mu_1\cdots\mu_n}(m_1\cdots m_n) = \int d\lambda\,\rho(\lambda)[p_{\mu_1}^1(m_1,\lambda)\cdots p_{\mu_n}^n(m_n,\lambda)]. \tag{5}$$

This is the most general form that accounts for the correlations by attributing them to some unspecified set of parameters λ common to all n particles, with distribution $\rho(\lambda)$, subject only to the requirement that the outcome at any one detector for given λ not depend on the choice of component (x or y) to be measured at any of the others. The representation (5) is generally regarded as the hallmark of a local theory that accounts for the correlations entirely in terms of information jointly available to the particles when they left their common source. It is the form tested by Bell's inequality and all of its generalizations. [It is also mathematically equivalent to the condition that there exists a single joint distribution of $2n$ variables, $P_{x_1y_1\cdots x_ny_n}$, that returns all the measured distributions as marginals. It is this mathematical fact that underlies efforts (such as Stapp's[9]) to derive nonlocality without invoking the parameters λ, and that explains why both approaches always give the same mathematical conditions on the observed correlation functions.]

If a representation (5) exists, then the mean of a product of x or y components of the spins of all the particles will be given by

$$E_{\mu_1\cdots\mu_n} = \int d\lambda\,\rho(\lambda)E_{\mu_1}^1(\lambda)\cdots E_{\mu_n}^n(\lambda), \tag{6}$$

where

$$E_\mu^j(\lambda) = p_\mu^j(\uparrow,\lambda) - p_\mu^j(\downarrow,\lambda). \tag{7}$$

In particular, the linear combination of such experimentally measured correlation functions corresponding to the linear combination of correlation functions whose theoretical value is given by the expansion (3) of (2) is evidently just

$$F = \int d\lambda\,\rho(\lambda)\frac{1}{2i}\left[\prod_{j=1}^n(E_x^j + iE_y^j) - \prod_{j=1}^n(E_x^j - iE_y^j)\right]. \tag{8}$$

We have noted that according to the quantum theory, for n particles in the state $|\Phi\rangle$, F is given by

$$F = \langle\Phi|A|\Phi\rangle = 2^{n-1}. \tag{9}$$

There is, however, a much more stringent bound on the size of any quantity that can be expressed in the form (8). Each of the $2n$ quantities E_x^j, E_y^j appearing in the integrand of (8) is constrained by (7) to lie between -1 and 1. Since the integrand of (8) is linear in each E_μ^j (holding the other $2n-1$ of them fixed), it assumes its extremal values at the boundaries of each of their domains. It is therefore everywhere bounded by the largest of the extremal values it assumes at the points where each of the E_x^j and E_y^j is independently taken to be $+1$ or -1. Since (8) can also be written as

$$F = \mathrm{Im}\left[\int d\lambda\,\rho(\lambda)\prod_{j=1}^n(E_x^j + iE_y^j)\right], \tag{10}$$

at the extremal points F is just the imaginary part of an average of a product of complex numbers each of which has magnitude $\sqrt{2}$ and phase $\pm\pi/4$ or $\pm 3\pi/4$. When n is even the product can lie along the imaginary axis and attain a maximum value of $2^{n/2}$; when n is odd the product must be at 45° to the imaginary axis and its imaginary part can only attain the maximum value $2^{(n-1)/2}$. Thus if F can be represented in the form (8) then

$$\begin{aligned}&F \le 2^{n/2}, \quad n \text{ even},\\ &F \le 2^{(n-1)/2}, \quad n \text{ odd}.\end{aligned} \tag{11}$$

When n exceeds 2 either bound is less than the quantum theoretic value (9) which exceeds it by the exponentially large factor of $2^{(n-2)/2}$ (for n even) or $2^{(n-1)/2}$ (for n odd).

Note that the large-n GHZ state (1) is a superposition of two states that differ in all n degrees of freedom: Only the mean values of n-particle operators can reveal interference effects. It thus combines two of the most peculiar features of the quantum theory. It displays an extreme form of quantum nonlocality as a direct manifestation of interference effects between macroscopically distinct states.

This work was supported in part by the National Science Foundation, Grant No. DMR 8920979.

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