Universal Scaling of the Stress Field at the Vicinity of a Wedge Crack in Two Dimensions and Oscillatory Self-Similar Corrections to Scaling

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We study the divergence of the stress field near the tip of a wedge crack of head angle between π and 2π. A new universal family of solutions is found, where the dominant singularity is characteristic of a pure tension on the external boundaries. Further from the tip the first correction is a power-law characteristic of a pure shear on those boundaries. When the head angle is in a certain range, higher-order modifications to the field are found to be periodic in the logarithm of the distance from the tip. Inside this range oscillatory solutions appear as corrections to the power-law behavior. The relevance of these solutions to sidebranching and to self-similar pattern formation of cracks is discussed.

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The mechanics and dynamics of stable crack propagation have been under study for a very long time. Recent developments in the study of pattern formation, on the one hand, and in industrial applications, on the other, led to an increase in the efforts invested in these problems. It is well established experimentally[1-5] that in ductile materials, under physically important boundary conditions, there often exists an extended regime of stable crack growth which intervenes between the initiation of a crack and its final instability. There also occur in nature processes of cracking without reaching the final catastrophic regime, such as stress corrosion cracking, the cracking of particle arrays,[6] and even of a muddy landscape upon drying. Simple lattice models of such quasistatic crack propagation, i.e., governed by a quasistatic elastic stress field, have recently been studied in two dimensions (2D) by computer simulation.[7,8] In these models the cracks are free to tip split and sidebranch and the resulting crack patterns have been interpreted as being spatially self-similar.

We present what we believe to be the first analytic results that can lead to formation of such self-similar quasistatic crack growth. We relate to these simulations in assuming that the propagation of each crack branch is controlled uniquely (and in an idealized way) by the concentration of the stress field at its tip. However, our results are derived for a continuum 2D isotropic medium, rather than on a lattice.

We demonstrate the instability of the planar "crack," which bears striking resemblance to the equivalent result in diffusion-limited aggregation (DLA).[9] As a result, we are led to consider a conical (or wedge) approximation to highly ramified growth and can make tentative predictions concerning its selected apex angle (for statistical isotropic growth). We find *complex* exponents for the corrections to scaling in the stress concentration around a wedge model of a crack envelope, indicating that the corrections are oscillatory in the logarithm of distance from the tip. This may provide an explicit mechanism for the selection of the geometrical approximate periodicity which must develop if the crack morphology is to evolve into a self-similar form.

For isotropic medium our results are universal in that Poisson's ratio is an irrelevant parameter.[10] This universality of crack propagation problems in two dimensions, which was seen in simulations,[7] may be understood by representing the stress tensor as $\sigma_{xx}=\partial_{yy}\Phi$, $\sigma_{yy}=\partial_{xx}\Phi$, and $\sigma_{xy}=-\partial_{xy}\Phi$, where the scalar function Φ is known as the Airy stress function (ASF). For an isotropic medium (of arbitrary elastic constants) Φ obeys the biharmonic equation $\nabla^4\Phi=0$. Taking the surface of a growing crack to bear no *applied* force, it may be shown that $\nabla\Phi$ is a constant along the crack surface. Since only second-order derivatives in Φ enter the stress, we are at liberty to choose that the crack surface has the particular value $\nabla\Phi=0$, and then further that $\Phi=0$, reducing the boundary conditions to $\Phi=\partial_n\Phi=0$.

At the wedge boundary there exists in principle an arbitrary relation between the local advance velocity, or breakage rate, v and the stress at the surface. However, in 2D at an unloaded boundary, only the normal stress in the tangential direction can be nonzero, and so we may take $v=f(\nabla^2\Phi)$. The cases where $f(x)\sim|x|^\eta$ will be of particular interest.

We begin by noting the instability of a planar front propagating into the medium: This might represent either a front of stress corrosion, or a planar envelope of many microscopic crack tips. The unperturbed solution consists of a flat boundary along, say, $y=0$ with an ASF in $y>0$, $\Phi(x,y)=\sigma_0 y^2/2$, leading to steady advance of the front upwards with $v=f(\sigma_0)$. We now consider the boundary perturbed to

$$y=Y(x)=\sum_k Y_k e^{ikx}\,, \qquad (1)$$

with $Y_0=0$ at $t=0$ and the other Y_k small. To first order in Y_k the appropriate solution of the ASF is

$$\Phi(x,y)=\tfrac{1}{2}\sigma_0 y^2-\sigma_0\sum_k Y_k y\exp[ikx-|k|y]\,, \qquad (2)$$

from which we can compute the advance rate along the interface. To first order in Y_k we find that their growth relative to the advance of the mean interface is given by

$$\frac{dY_k}{dY_0}=2\eta_{\text{eff}}|k|Y_k, \quad \text{where } \eta_{\text{eff}}\equiv\left[\frac{d\ln f(\sigma)}{d\ln\sigma}\right]_{\sigma=\sigma_0}. \tag{3}$$

This exponential growth of surface corrugations differs from diffusion-limited growth controlled by a Laplace field, only by the factor of 2 in (3). It is well known that in the latter problem and with a sharp spatial cutoff on the allowed modes of growth (for example, an underlying particle size or lattice spacing), the growth front ramifies to form a highly branched, self-similar fractal structure.[11] We thus entertain the possibility that, as simple lattice-based computer simulations have suggested,[7,8,11] the same applies in our present problem.

For a self-similar structure to develop we anticipate power-law singularities in the stress concentration at the leading crack tips. In the presence of competing side-branches the conventional single straight-line crack is not a good model from which to calculate this stress concentration. We study instead a wedge envelope on the grounds that (i) the stressless interior is a reasonable representation of highly cracked material; (ii) this is the simplest scale-invariant shape with a tip feature; and (iii) it gives an expected power-law singularity in the stress concentration with exponent related to a geometric feature—in this case, the wedge angle.

Consider a crack embedded in an otherwise homogeneous and isotropic medium, as depicted in Fig. 1. This crack is assumed to have the shape of a wedge whose head angle is $0<2(\pi-\alpha)\le\pi$. Throughout our analysis we assume that the crack is static or, at most, propagating adiabatically, such that at each instant the stress field around it can be considered at equilibrium. This assumption excludes from our consideration brittle fractures whose typical propagation time is shorter than the time it takes the field to relax to equilibrium. Assuming that the ASF can be decomposed over powers of r, the radius from the tip, it is straightforward to find solutions to the biharmonic equation of the form

$$\Phi_{\text{even}}=\sum_m[p_m\cos(m+1)\theta+q_m\cos(m-1)\theta]r^{m+1}, \tag{4}$$

which are even functions of Φ, with analogous odd solutions having the cosine term replaced by a sine term. The allowed values of m are in general complex, and together with the ratio of the coefficients p_m/q_m these are selected by the wedge boundary conditions, $\Phi=\partial_\theta\Phi=0$ for $\theta=\pm\alpha$, leading to

$$(\sin 2\alpha)/2\alpha\pm(\sin 2m\alpha)/2m\alpha=0 \tag{5}$$

for the even (+) and odd (−) solutions, respectively. The odd version of (5) coincides with the one obtained for a similar problem with load applied to an elastic wedge.[12] However, unlike in that case, we will see below that it is the even version that yields the dominant singularity near the tip, while the solution to the odd equation only modifies this behavior further away. In the following we call m_{odd} and m_{even} the smallest solutions for m for the odd and even versions of (5), respectively.

Since the first term on the left-hand side of (5) is real, then the imaginary part of the second should vanish, yielding a relation between $\xi\equiv\text{Re}(2m\alpha)$ and $\eta\equiv\text{Im}(2m\alpha)$. This relation is realized on three families of lines in the ξ-η plane: (i) the "primary branch" $\eta=0$; (ii) the "secondary branch" where $\tanh\eta/\eta=\tan\xi/\xi$; (iii) $\xi=0$, $\eta\neq0$. The value of $\sin(2m\alpha)/2m\alpha$ on these lines is, respectively, (i) $\sin\xi/\xi$, (ii) $\cosh\eta(\xi)\sin\xi/\xi$, where $\eta(\xi)$ obeys the above relation along the secondary branch, and (iii) $\sinh\eta/\eta$. Case (iii) leads to no solutions. To have a physical solution to (5) we require that the displacement field is single valued; this leads to the restriction $\text{Re}(m)=\xi/2\alpha>0$.

Near the tip it is the strongest allowed singularity [i.e., the smallest $\text{Re}(m)>0$ that solves (5)], which dominates the field's behavior. The strongest values of the

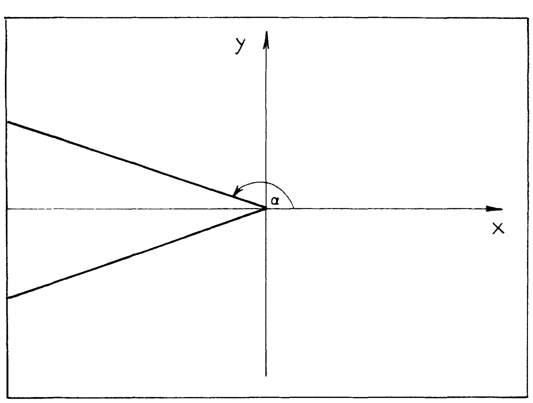

FIG. 1. A crack whose envelope is modeled by a wedge. At $y\to\pm\infty$, a combination of extension and shear are applied to the boundaries.

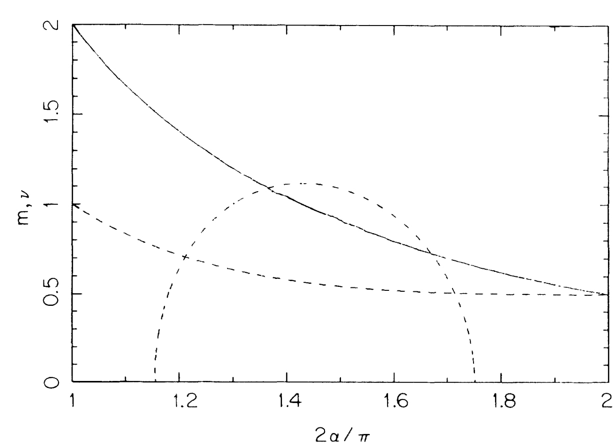

FIG. 2. The lowest even- (dashed line) and odd- (solid line) order solutions for m in Eq. (5), plotted against 2α. The dash-dotted line represents the frequency ν of the lowest-order oscillatory solution, as a function of 2α.

odd and even values, $m_{\rm odd}$ and $m_{\rm even}$, are real and are shown in Fig. 2. Except at the end points $(2\alpha=\pi,2\pi)$ $m_{\rm even}<m_{\rm odd}$, implying that the tension at the far boundaries is more important than the shear to the stress near the wedge tip, $\sigma\sim r^{m_{\rm even}-1}$. For $2\alpha<2\alpha_0\approx 1.43\pi$, $m_{\rm odd}=1$ indicating that the odd contribution is nonsingular.

We next demonstrate a possible application of the above results, which are also of direct engineering significance, to the stability of a crack pattern under growth, with respect to competition amongst its major branches. In an earlier paper[13] one of us gave an argument for the stability of a diffusion-limited growth of N equivalent major arms (each of which could be highly ramified) with respect to spontaneous modulation of the arm length from R to $R+\delta R\cos S\theta$. We now extend that argument to the present problem. Starting from the ASF for the unmodulated growth,

$$\Phi(r,\theta)=R\sum_{n=1}^{\infty}[a_n+(r/R)^2b_n]\cos n\theta+R[a_0+(r/R)^2b_0]\ln(r/R)\,, \tag{6}$$

and assuming that near each tip $\Phi(R+\rho)\sim\rho^{1+m}$, we find that the modified ASF satisfies

$$\Phi_{\rm new}(R+\delta R\cos S\theta+\rho)=\Phi(R+\rho)[1+(S-1)(1+m_{\rm even})\cos S\theta\,\delta R/R]\,. \tag{7}$$

In a model where the velocity of the tip advance obeys $v\sim\sigma^{\eta}$,[7,8] this then gives $\delta R/R$ increasing with growth for $\eta(S-1)(1+m_{\rm even})>1$. Identifying $S=N/2$ as the highest sustainable mode of modulation, then the maximal stable number of arms obeys

$$N^*/2=1+1/\eta(1+m)\,, \tag{8}$$

which, since $\frac{1}{2}\le m\le 1$, can be bounded by $2+1/\eta\le N^*\le 2+4/3\eta$. Noninteger N^* are probably best interpreted in terms of $2\pi/N^*$ being the minimal stable angle between (the direction of growth of) neighboring major arms.

This argument can now be complemented by relating N^* to the tip angle $2(\pi-\alpha)$ to yield, from (8), the selected value of α^* (and hence of $m_{\rm even}$). A crude estimation can be obtained by assuming that adjacent tips have common edges, the envelopes then closing to form a polygon of N^* sides of integer N^*. This gives $2\alpha^*=\pi(1+2/N^*)$, so that if we assume that the pattern evolves at marginal stability, we predict selection of α^* that satisfies

$$2\pi-2\alpha^*=\pi/\{1+[1+m(\alpha^*)]\eta\}\,. \tag{9}$$

We now turn to the higher-order corrections of (5) corresponding to subdominant singularities in the ASF. We find that, unlike the leading term, these can be oscillatory, which may be significant in determining the self-similar shape of the cracks pattern. Figure 3 shows how these may be found graphically (see caption). In general, there is a finite number of real solutions for m, followed at larger Re(m) by an infinite sequence of complex solutions. As $F(2\alpha)\equiv\sin 2\alpha/2\alpha$ approaches zero the real solutions increase in number and they approach the set of half-integer values, whereas for $F(2\alpha)>0.125\ldots$ only the two leading solutions displayed in Fig. 2 remain real. Complex $m=\mu+i\nu$ corresponds to oscillatory terms in the ASF varying as $r^{\mu}\cos[\nu\ln(r/r_0)]$. A natural periodicity in ln$r$ of the crack pattern may be selected by our periodic ASF solutions; these correspond to "free" (spatial) oscillations in the sense that they can be sustained near the tip without corresponding oscillatory terms in the boundary conditions to derive them. Although we do not have a complete theory of the selection of spacing of major sidebranches, we anticipate that the selection may be strongly biased by resonant amplification of corrugations in the crack envelope at spatial frequencies (in ln$r$) matched to these modes. Moreover, because of the power-law suppression (μ) of increasingly high frequencies (ν), we expect the lowest modes to manifest most in real patterns.

Periodicity in ln$r$ is usually the fingerprint of self-similar patterns; such structures are invariant under

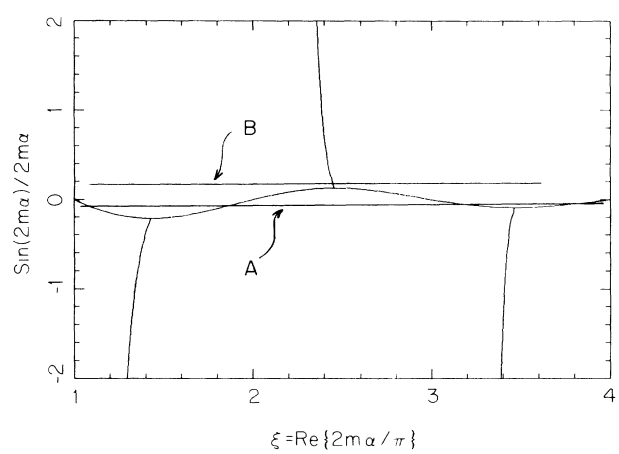

FIG. 3. The values of $\sin(2m\alpha)/2m\alpha$, when it is real, as a function of $\xi=\text{Re}(2m\alpha)$. The primary branch, when $\text{Im}(2m\alpha)=0$, is $F(\xi)=\sin\xi/\xi$. The lines of the secondary branch shoot off from the maxima, where $F(\xi_0)=\cos\xi_0$. An intersection of $A\equiv F(2\alpha)=\sin(2\alpha)/2\alpha$ with the primary (secondary) branch represents a real (complex) solution for m in the odd version of (5). Similarly $A\equiv -F(2\alpha)$ solves the even version. When $0.218\ge F(2\alpha)\ge 0.125$ [i.e., when 2α is within the range $(1.155\pi,1.75\pi)$, say, line B], there exist no real solutions for $\xi\ge 2\pi$, and the second- and higher-order corrections are all complex. As the value of F drops below the value of $F(\xi_0)$ at a local maximum, a complex solution disappears by splitting into two real roots, and the oscillatory correction is pushed to a higher order.

scale transformations and many of their properties follow a power law in the length scale. To the best of our knowledge this is the first exact solution that could explicitly select the formation of self-similar patterns in any aggregate. The wave number of the lowest mode in $\ln r$ space is shown (dash-dotted line) in Fig. 2 as a function of 2α. A way to test this mechanism of triggering the sidebranches formation is to inspect real aggregates of quasistable cracks and check that the wavelength, $2\pi/\nu$, and the head angle relate as we suggest.

Because of the independence of the solutions to the ASF of Poisson's ratio, the analysis presented here does not depend on the compressibility of the medium. This implies that these results are universal, applying to any 2D isotropic elastic system under the above boundary conditions. The difference between any two such systems can only stem from the different constitutive relations between the velocity v of the tip propagation and the magnitude of the local stress field it experiences, $v(r) = f(\nabla^2\Phi(r))$. Note, though, that the instability discussed above appears only when the function f increases with its argument. When the reverse is true the oscillatory solutions are suppressed and the crack smooths out.[14]

The present analysis assumes implicitly that (i) linear elasticity is applicable, which may fail for many types of cracks, and (ii) that there is no reclosure of cracks, thus excluding the boundaries from bearing stress across the surface. While the latter is intuitively very plausible, especially in an etching model, we cannot exclude this possibility in general. Any mechanism leading to such closure would be sensitive to the elastic displacements, and hence to the relative elastic constants.

It is hard to make a critical comparison of our results with the simulations of Refs. 7 and 8 because of their limited results. Although it is difficult to define precisely a number of selected arms from the simulations, the patterns obtained in Ref. 8 are consistent with our predicted trend of how N^* should decrease as η increases. It is interesting to note that this quantity, which we predict, appears to be independent of lattice details in the simulations, whereas the fractal dimension—for which we have been unable to arrive at any off-lattice prediction—does not appear to be universal. Perhaps we have yet to find the natural "masslike" measure for these objects.

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