Long-Time, Large-Scale Properties of a Randomly Stirred Compressible Fluid

Ilya Staroselsky, Victor Yakhot, Shigeo Kida, and Steven A. Orszag
Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544
(Received 26 March 1990)

The large-scale, long-time properties of a compressible fluid stirred by a Gaussian random force with correlation $\langle f_i f_j \rangle \propto k^{-y}$ are investigated. It is shown that when $\epsilon = 4+y-d > 3$ (d = space dimension) the effective sound velocity becomes scale dependent in the limit $k \to 0$ and the fluid obeys a universal equation of state. The effective Mach number is also scale dependent, reaching a fixed-point value $Ma^*(k) < 1$ when $k \to 0$. The predictions of the theory are compared with the results of direct numerical simulations.

PACS numbers: 47.25.Mr

The Navier-Stokes equation for an incompressible fluid stirred by the Gaussian random force defined by the pair-correlation function

$$\langle f_i(\mathbf{k},\Omega) f_j(\mathbf{k}',\Omega')\rangle \propto D_1 k^{-y} P_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta(\Omega+\Omega') \quad (1)$$
$$[P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2]$$

is a generic model describing important states of the fluid. First of all, when $y=-2$ and $D_1 = T\nu_0/\rho$ (T, ν_0, and ρ are the temperature, molecular viscosity, and density, respectively), the model describes both the static and dynamic properties of the fluid in thermodynamic equilibrium.[1-4] The case $y > -2$ corresponds to a nonequilibrium situation. When $y \to d$ (d being the space dimension), the velocity correlations generated by the Navier-Stokes equations with the forcing function (1) resemble qualitative and quantitative features of velocity correlations in fully developed turbulent flows and are characterized by the Kolmogorov energy spectrum. In the present work we are interested in the properties of a nonequilibrium *compressible* fluid stirred by a Gaussian random force similar to the force defined by (1) with $y > -2$. In principle, this theory can be developed in terms of a perturbation expansion in powers of Mach number $Ma_0 = v_{\rm rms}/c_{s0}$, where $v_{\rm rms}$ and c_{s0} are the characteristic velocity and sound speed, respectively. When $Ma_0 \to 0$, the development of the theory is relatively straightforward. In fully developed turbulence, $v_{\rm rms} \propto L^\alpha$ (L is the linear dimension of the system) with $\alpha > 0$, so that $v_{\rm rms} \to \infty$ in the infinite medium ($L \to \infty$) that we are interested in here. In this case the definition of Ma_0 given above is meaningless since $Ma_0 \to \infty$ and one can erroneously expect the dynamics of the flow to be dominated by shock waves. It is shown here that when $\alpha > 0$ the effective sound speed also grows with scale $l \propto k^{-1}$ and the relevant renormalized Mach number reaches a finite fixed point. The renormalization of the sound speed demonstrated below is the most striking, although easily understandable, result of this work. This effect is essential to understanding the physics of compressible flows since the appearance of the dimensional parameter c_{s0} makes simple dimensional considerations invalid.

Consider the momentum equation and the continuity equation for compressible flow:

$$\partial_t \mathbf{J} = -\nabla \frac{\mathbf{J}\cdot\mathbf{J}}{n} - c_{s0}^2 \nabla n + \nu_0 \nabla^2 \frac{\mathbf{J}}{n} + \left(\frac{\nu_0}{3}+\sigma_0\right)\nabla\left(\nabla\cdot\frac{\mathbf{J}}{n}\right) + \mathbf{f}, \quad (2)$$

$$\partial_t n = -\nabla\cdot\mathbf{J}, \quad n = \bar{n} + \delta n. \quad (3)$$

Here $\mathbf{J}$ is the momentum density and n is the fluid mass density, ν_0 and $\mu_0 \equiv \nu_0/3 + \sigma_0$ are the kinematic shear and bulk viscosities, respectively, and $c_{s0}^2 = \partial p/\partial n$ is the square of the sound velocity. The zero-mean Gaussian random stirring force on the right-hand side of (2) is defined by its correlation function

$$\langle f_i(\mathbf{k},\Omega) f_j(\mathbf{k}',\Omega')\rangle = (2\pi)^{d+1} 2D_0 k^{-y}\delta_{ij}\delta(\mathbf{k}+\mathbf{k}')\delta(\Omega+\Omega'). \quad (4)$$

Let us expand the momentum equation (2) in powers of $\delta n/\bar{n}$ and keep only low-order terms:

$$\partial_t u_i = -\partial_i u_j u_j [1-\delta n/\bar{n} + (\delta n/\bar{n})^2] - c_s^2 \partial_i \delta n + \nu \partial_j \partial_j u_i [1-\delta n/\bar{n} + (\delta n/\bar{n})^2] + \mu \partial_i \partial_j u_j [1-\delta n/\bar{n} + (\delta n/\bar{n})^2] + f_i. \quad (5)$$

Here $\mathbf{u} \equiv \mathbf{J}/\bar{n}$. Equation (5) is closed when we express the density fluctuations δn through the momentum fluctuations using (3). Finally, the Fourier-transformed momentum equation takes the form

$$G_{ij}^{-1}(\hat{k}) u_j(\hat{k}) = f_i + \frac{g_0}{2}\int \frac{d^{d+1}\hat{q}_1}{(2\pi)^{d+1}} \frac{d^{d+1}\hat{q}_2}{(2\pi)^{d+1}} \Gamma_{imn}(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2)\delta(\hat{k}+\hat{q}_1+\hat{q}_2) u_m(\hat{q}_1) u_n(\hat{q}_2)$$
$$+ \frac{g_0^2}{6}\int \frac{d^{d+1}\hat{q}_1}{(2\pi)^{d+1}} \frac{d^{d+1}\hat{q}_2}{(2\pi)^{d+1}} \frac{d^{d+1}\hat{q}_3}{(2\pi)^{d+1}} \Gamma_{imns}(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)\delta(\hat{k}+\hat{q}_1+\hat{q}_2+\hat{q}_3) u_m(\hat{q}_1) u_n(\hat{q}_2) u_s(\hat{q}_3), \quad (6)$$

where $\hat{k}=(\mathbf{k},\omega)$ and g_0 is a formal expansion parameter which will be set equal to unity at the end of the calculation. The tensorial Green's function

$$G^0_{ij}=G^0_\perp P_{ij}(k)+G^0_\parallel R_{ij}(k)\quad [R_{ij}(\mathbf{k})=k_ik_j/k^2] \tag{7}$$

is given by

$$G^0_\perp=(-i\omega+\nu_0k^2)^{-1},\quad G^0_\parallel=[i(\omega+c_{s0}^2k^2/\omega)+\lambda_0k^2]^{-1}\quad (\lambda_0\equiv\nu_0+\mu_0)\,. \tag{8}$$

The vertexes $\Gamma_{lmn}(k,k_1,k_2)$ and $\Gamma_{lmns}(k,k_1,k_2,k_3)$ are defined by

$$\Gamma_{lmn}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)=-ik_m\delta_{ln}-ik_n\delta_{lm}+\frac{k_{2n}}{\omega_2}(\delta_{lm}\nu k^2+\mu k_lk_m)+\frac{k_{1m}}{\omega_1}(\delta_{ln}\nu k^2+\mu k_lk_n)\,, \tag{9}$$

$$\begin{aligned}\Gamma_{lmns}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=&\,ik_m\delta_{ln}\frac{k_{3s}}{\omega_3}+ik_m\delta_{ls}\frac{k_{2n}}{\omega_2}+ik_n\delta_{lm}\frac{k_{3s}}{\omega_3}+ik_n\delta_{ls}\frac{k_{1m}}{\omega_1}+ik_s\delta_{ln}\frac{k_{1m}}{\omega_1}+ik_s\delta_{lm}\frac{k_{2n}}{\omega_2}\\&-\delta_{ln}\nu k^2\frac{k_{3s}}{\omega_3}\frac{k_{1m}}{\omega_1}-\delta_{lm}\nu k^2\frac{k_{2n}}{\omega_2}\frac{k_{3s}}{\omega_3}-\delta_{ls}\nu k^2\frac{k_{2n}}{\omega_2}\frac{k_{1m}}{\omega_1}\\&-\mu k_lk_m\frac{k_{2n}}{\omega_2}\frac{k_{3s}}{\omega_3}-\mu k_lk_n\frac{k_{1m}}{\omega_1}\frac{k_{3s}}{\omega_3}-\mu k_lk_s\frac{k_{2n}}{\omega_2}\frac{k_{1m}}{\omega_1}\,.\end{aligned} \tag{10}$$

The problem (6)-(10) is formulated on the interval $|k|<\Lambda_0$, where Λ_0 is the wave number beyond which dissipation takes place. The details of the scale-elimination procedure are essentially the same as in the case of an incompressible fluid discussed in Refs. 2-4. We eliminate modes $u^>(k)$ with wave vectors satisfying $\Lambda_0e^{-r}<k<\Lambda_0$ from the equations of motion for the equations for the slow modes $u^<(k)$ with wave vectors from the interval $k<\Lambda_0e^{-r}$. We keep only the terms up to order g_0^2 and average over the fast modes $u^>(k)$. The resulting equation for the slow modes $u^<(k)$ is

$$\begin{aligned}u_i^<(\hat{k})=&\,G_{il}(\hat{k})f_l+g_0G_{il}(\hat{k})\int^>\frac{d^{d+1}\hat{q}}{(2\pi)^{d+1}}\Gamma_{lmp}(-\mathbf{k},\mathbf{k}/2+\mathbf{q},\mathbf{k}/2-\mathbf{q})\Gamma_{tns}(\mathbf{k}/2-\mathbf{q},\mathbf{k}/2+\mathbf{q},-\mathbf{k})\\&\times G_{pt}(\hat{k}/2-\hat{q})2D_0(\hat{q}+\hat{k}/2)^{-y}G_{mp}(\hat{q}+\hat{k}/2)G_{np}(-\hat{q}-\hat{k}/2)G_{sj}(\hat{k})u_j^<(\hat{k})\\&+g_0^2G_{il}(\hat{k})\int^>\frac{d^{d+1}\hat{q}}{(2\pi)^{d+1}}\Gamma_{lmns}(\mathbf{k},-\mathbf{k},\mathbf{q},-\mathbf{q})2D_0k^{-y}G_{mp}(\hat{q})G_{np}(-\hat{q})G_{sj}(\hat{k})u_j^<(\hat{k})\\&+\frac{g_0}{2}G_{il}(\hat{k})\int\frac{d^{d+1}\hat{q}_1}{(2\pi)^{d+1}}\frac{d^{d+1}\hat{q}_2}{(2\pi)^{d+1}}\Gamma_{lmn}(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2)\delta(\hat{k}+\hat{q}_1+\hat{q}_2)u_m^<(\hat{q}_1)u_n^<(\hat{q}_2)\\&+\frac{g_0^2}{6}G_{il}(k)\int\frac{d^{d+1}\hat{q}_1}{(2\pi)^{d+1}}\frac{d^{d+1}\hat{q}_2}{(2\pi)^{d+1}}\frac{d^{d+1}\hat{q}_3}{(2\pi)^{d+1}}\Gamma_{lmns}(\mathbf{k},\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)\delta(\hat{k}+\hat{q}_1+\hat{q}_2+\hat{q}_3)u_m^<(\hat{q}_1)u_n^<(\hat{q}_2)u_s^<(\hat{q}_3)\,,\end{aligned} \tag{11}$$

where the symbol $\int^>$ means integration over the band $\Lambda_0e^{-r}<k<\Lambda_0$. Equation (11) is defined on the domain $0<k<\Lambda_0e^{-r}$. The contributions involving $\int^>$ in the equation for the large-scale modes $u^<(\hat{k})$ are associated with corrections to viscosities and sound velocity for (2) defined on the domain $0<k<\Lambda_0$. Thus these terms take into account the role of the eliminated modes on the dynamics of the remaining modes $u^<(\hat{k})$.

First, let us consider the case of weak compressibility. In the incompressible case[3,4] ($\mathrm{div}\mathbf{J}/n=0$), the dynamics of velocity fluctuations generated by the random force is characterized in the scaling regime by the characteristic frequency $\omega=O(D_0^{1/3}k^{2-\epsilon/3})$, $\epsilon=4+y-d>0$. Thus the maximum wave number at which various dissipation overcomes nonlinearity can be found from the relation $D_0^{1/3}k_d^{-\epsilon/3}\propto\nu_0$ so that $\omega_{\max}\propto D_0^{1/3}k_d^{2-\epsilon/3}$. The frequency integration in $\int^>$ of (11) should be carried out over the domain $-\omega_{\max}<\omega<\omega_{\max}$. It will be shown below that in the weakly compressible limit $c_{s0}k_d\gg\omega_{\max}$ the correction to the propagator due to the potential component of the forcing function is $O(Ma^8)$ so that the contributions $O(kv(k))$ can be neglected when $Ma\to0$. In this limit the renormalized propagator is $G_\perp^{-1}=-i\omega+\nu_1k^2$, $G_\parallel^{-1}=i(\omega+c_{s1}^2k^2/\omega)+\lambda_ck^2$, where $\nu_1=\nu_0[1+A_d\bar{g}_0^2(e^{\epsilon r}-1)/\epsilon]$. The dimensionless coupling constant is $\bar{g}_0=g_0(D_0/\nu_0^3\Lambda_0^\epsilon)^{1/2}$, and the renormalized viscosity is $\lambda_c\equiv\nu_c+\mu_c=\lambda_0+\delta\lambda$, where

$$\delta\lambda\propto\int^>\int_0^{\omega_{\max}}\frac{d\Omega\,d^dq}{2(\nu_0+\mu_0)q^2[(\Omega+c_s^2q^2/\Omega)^2+(\nu_0+\mu_0)^2q^4]^2}\,. \tag{12}$$

When $c_{s0}\to\infty$ and $|q|\propto O(k_d)$, $\delta\lambda_p\propto Ma^8$. In this case the energy spectrum of the flow is

$$E=E_{\mathrm{inc}}+E_{\mathrm{comp}}=E_{\mathrm{inc}}+O(Ma^8)\,, \tag{13}$$

where

$$E_{\rm inc} \propto {\rm Tr} \int_0^{\omega_{\max}} D_0 k^{-y} \frac{P_{ij}(k)}{\omega^2+\nu^2 k^4} d\omega, \quad E_{\rm comp} \propto {\rm Tr} \int_0^{\omega_{\max}} D_0 k^{-y} \frac{R_{ij}(k)}{(\omega+c_s^2 k^2/\omega)^2+\lambda^2 k^4} d\omega . \tag{14}$$

In deriving this result we did not consider the sound-speed renormalization since the frequency integration (12) enters the corrections to the sound velocity and thus $\delta c_s \propto Ma^8$ which does not change the estimate (13).

In the second limiting case of finite Ma we take $\omega_{\max}/c_s k_d \rightarrow \infty$ and thus the frequency integration is to be carried out over the domain $-\infty < \omega < \infty$. In this case, calculation of the integrals in (11) with the functions (7)-(10) leads to corrections to the bare values of the viscosities ν,μ, and the speed of sound c_s^2,

$$\delta\nu = A\nu M_1(\theta,R)C, \quad \delta\lambda = A\lambda M_2(\theta,R)C, \quad \delta c_s^2 = Ac_s^2 M_3(\theta,R)C . \tag{15}$$

Here $A=1/30\pi^2$, $C=\bar{g}^2(e^{\epsilon r}-1)/\epsilon$, and we introduce the dimensionless functions $\bar{g}^2 = D_0\Lambda_c^{-\epsilon}/\nu^3$, $\theta=\lambda/\nu$, and $R=\nu^2\Lambda_c^2/c_s^2$. The functions M_1, M_2, and M_3 are, in general, functions of ϵ. When $\epsilon \rightarrow 0$ and $r \rightarrow \infty$,

$$M_1 = 3-\tfrac{1}{2}\epsilon + Rm_1(\theta,R), \quad M_2 = 4-\tfrac{7}{8}\epsilon + Rm_2(\theta,R), \quad M_3 = Rm_3(\theta) , \tag{16}$$

where the functions m_1, m_2, and m_3 are $O(1)$ and $R\rightarrow 0$ (see below). In this case we find from (15) and (16)

$$\nu = 1.22\epsilon^{-1/3}D_0^{1/3}\Lambda_c^{-\epsilon/3}, \quad \lambda = 1.62\epsilon^{-1/3}D_0^{1/3}\Lambda_c^{-\epsilon/3}, \quad c_s = c_{s0} , \tag{17}$$

and the effective coupling constant $\bar{g}^2 = o(\epsilon) \rightarrow 0$. This is the limit of incompressible flow. The renormalized Mach number $M^2(r) \propto \epsilon^{1/3} c_{s0}^{-2} D_0^{-1/3} \exp[2(1-\epsilon/3)r]$ goes to zero on large scales ($r\rightarrow\infty$) and the large-scale flow can be described in the incompressible limit.

Another interesting case is $\epsilon \rightarrow 4$. The functions M evaluated for $\epsilon=4$ are

$$M_1 = 1+2R\theta^{-1}+\tfrac{9}{8}R-\tfrac{5}{2}\theta^{-1}(\theta-1)^2R^2, \quad M_2 = \tfrac{1}{2}+\tfrac{57}{8}R\theta^{-1}+\tfrac{17}{8}R-\tfrac{5}{2}\theta^{-1}(\theta-1)^3R^2,$$
$$M_3 = -\tfrac{5}{8}R\theta^{-1}+\tfrac{5}{4}R . \tag{18}$$

By variation of the dissipation cutoff $\Lambda_c \equiv \Lambda_0 e^{-r}$ and using (18) we obtain the differential form of the recursion relations for the dimensionless functions $\bar{g}^2$, θ, and R:

$$d\bar{g}^2(r)/dr = \epsilon\bar{g}^2 - 3AM_1(\theta,R)\bar{g}^4, \quad d\ln\theta(r)/dr = A\bar{g}^2[M_2(\theta,R)-M_1(\theta,R)] ,$$
$$d\ln R(r)/dr = -2+2AM_1(\theta,R)\bar{g}^2 - AM_3(\theta,R)\bar{g}^2 . \tag{19}$$

In the limit $r\rightarrow\infty$ the renormalized nonlinear coupling parameter $\bar{g}$ goes to the fixed point $\bar{g}=(\epsilon/3AM_1)^{1/2}$. In the vicinity of this fixed point

$$d\ln R(r)/dr = -2+\tfrac{2}{3}\epsilon - AM_3(\theta,R)\bar{g}^2 , \tag{20}$$

with $M_3(\theta,R)>0$. We can see that when $\epsilon<3$ the dimensionless ratio $R\rightarrow 0$. When $\epsilon>3$, $R\rightarrow$ const >0. When $\epsilon=4$ we find from (18) and (19) that $\theta\rightarrow 1.5$, $R\rightarrow 3.25$, and

$$\nu = 2.14\left[\frac{2D_0S_d}{(2\pi)^d}\right]^{1/3}\Lambda_c^{-4/3}, \quad \lambda = 3.22\left[\frac{2D_0S_d}{(2\pi)^d}\right]^{1/3}\Lambda_c^{-4/3}, \quad c_s = 3.84\left[\frac{2D_0S_d}{(2\pi)^d}\right]^{1/3}\Lambda_c^{-1/3}. \tag{21}$$

The energy spectrum $E(k)$ can be calculated in the lowest order in ϵ from the equation $u_i = G_{ij}f_j$, where G_{ij} is the renormalized Green's function (7),(8) evaluated with the functions $\nu(k)$, $\lambda(k)$, $c_s^2(k)$ given by (21) with $\Lambda_c = k$. If we take into account only the terms up to g^2, the correlation function of velocity fluctuations corresponding to $\epsilon=4$ is

$$U_{ij}(\mathbf{k},\omega) = (2\pi)^{d+1}2D_0k^{-d}\left[\frac{P_{ij}(k)}{\omega^2+\nu(k)^2k^4} + \frac{R_{ij}(k)}{(\omega+c_s^2k^2/\omega)^2+\lambda(k)^2k^4}\right] . \tag{22}$$

The energy spectrum is

$$E(k) = E_{\rm inc}(k)+E_{\rm comp}(k) \propto D_0^{2/3}k^{-5/3}$$

with the functions $E_{\rm inc}(k)$ and $E_{\rm comp}(k)$ defined by formulas (14) and $E_{\rm inc}/E_{\rm comp}=3$. Terms higher order in $\delta n/\bar{n}$ neglected in the derivation of the equations of motion (6) are all of higher order in the ϵ expansion and are omitted as irrelevant.

Two central predictions of the present theory are (1) $E_{\rm inc}/E_{\rm comp}$ = const, and (2) the effective sound velocity is scale dependent, $c_s(L) \propto L^{1/3}$. To test these predictions, direct numerical simulations of the compressible, adiabatic Navier-Stokes equations driven by the random

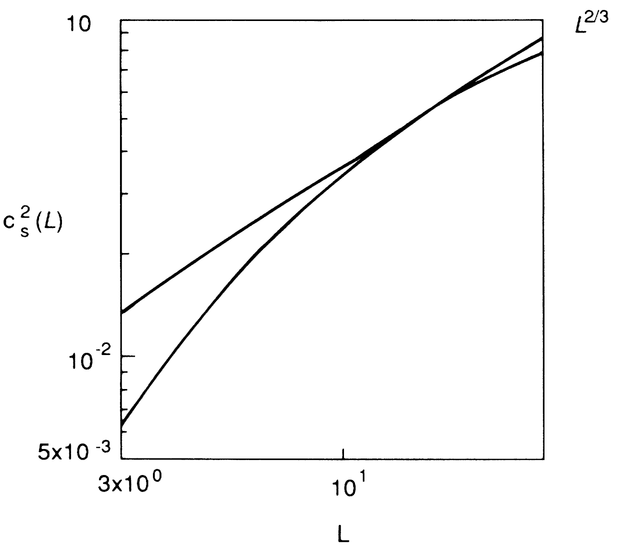

FIG. 1. Variance of speed of sound in a randomly stirred compressible turbulent flow as a function of length scale. The Mach number in the computation $Ma_0=1$; Reynolds number based on Taylor microscale $R_\lambda=20$.

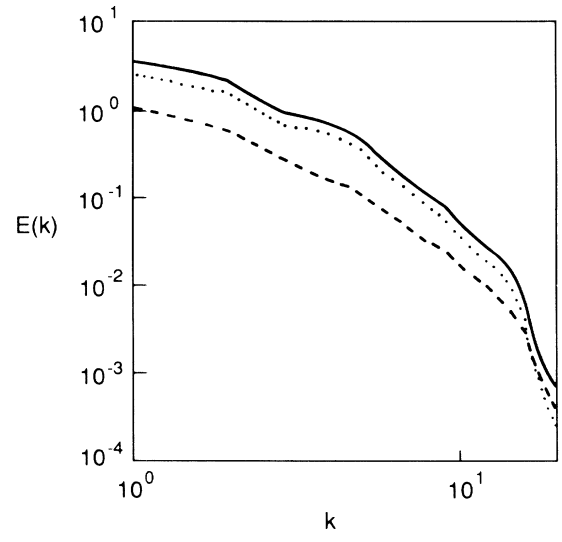

FIG. 2. Energy spectrum $E(k)$ in compressible flow. (——) $E_{\rm tot}$; (· · ·) $E_{\rm inc}$; (---) $E_{\rm comp}$.

force (4) with $y=d$ were performed. A spectral code with 64^3 mesh points was used.[5] The Reynolds number based on the Taylor microscale was $R_\lambda=20$. Plots of the energy spectra $(E_{\rm comp}, E_{\rm inc})$ and the variance of the speed of sound calculated as a function of length scale are presented in Figs. 1 and 2. To analyze the speed of sound $c_s^2(L)$, the expression $c_s^2=\gamma\rho^{\gamma-1}=\partial p/\partial\rho$ ($\gamma=1.4$) is averaged over boxes of linear dimension L. We see that the sound velocity depends strongly on the length scale according to the prediction of the present theory although the relatively low Reynolds number of the numerical experiment does not allow unambiguous verification of the scaling exponent.

The effect of sound-speed modification due to turbulent velocity fluctuations has been discussed by Chandrasekhar[6] and Bonazzola *et al.*[7] in the context of turbulence in the Jeans instability of a self-gravitating gas. Here we have investigated the effect of the spectrum of turbulent velocity fluctuations on sound-speed renormalization and derived for the first time a scale-dependent sound velocity. Application of the present results to astrophysical problems will be the subject of future communications.

This work was supported by the Office of Naval Research, the U.S. Air Force, and NASA.

[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1982).

[2] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[3] V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1986); Phys. Rev. Lett. **57**, 1722 (1986).

[4] W. P. Dannevik, V. Yakhot, and S. A. Orszag, Phys. Fluids **30**, 2021 (1987).

[5] S. Kida and S. A. Orszag (to be published).

[6] S. Chandrasekhar, Proc. Roy. Soc. London A **210**, 18 (1951).

[7] S. Bonazzola, M. Perault, J. L. Puget, J. Heyvaerts, and E. Falgarone (private communication).