

## Generation of Structure on a Cosmic-String Network

B. Allen and R. R. Caldwell

*Department of Physics, University of Wisconsin at Milwaukee, P.O. Box 413, Milwaukee, Wisconsin 53201*

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When cosmic strings cross and intercommute, four kinks are created. We calculate the linear density of these kinks,  $K(t)$ , in an expanding universe. After a period of rapid initial growth,  $K(t)$  approaches the scaling  $K(t) \propto t^{-1}$ . However, due to the slow decay of kinks, the kink density is orders of magnitude larger than one might expect. Thus, we predict that a single horizon-length segment should have  $\approx 10^6$  kinks in the radiation-dominated era. This may explain the lack of scaling behavior in the formation of loops observed in numerical simulations.

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In this paper, we consider the formation of kinks on a network of cosmic strings in an expanding, radiation-dominated universe.<sup>1</sup> A kink is a discontinuity in the derivative of the tangent vector along the string—the direction of the string changes abruptly at such a point.<sup>2</sup> Of course, the discontinuity is not present on the smallest length scales, and if  $\eta$  is the characteristic radius of the core of the string (typically  $10^{-29}$  cm), then the kink has a comparable radius. One can show that these kinks move along the characteristics of the wave equation (at the speed of light) so that if the transverse velocity of the string is  $\mathbf{v}$ , then the longitudinal velocity of the kink is  $(1 - v^2)^{1/2}$ .

The initial string network is formed with no kinks. During the subsequent evolution of the string network, kinks are created by the intercommutation (i.e., crossing and rejoining) of two strings (or two distant parts of the same string). Immediately after an intercommutation, the string segments lying on either side of the crossing point will have different velocities, and thus a kink is present. Since a kink moves away in both directions from the point of formation, each of the intercommuted strings acquires two kinks, and the total number of kinks on the network is increased by four.

In order to estimate the number of kinks on the infinite-string network, we adopt the “one-scale” model of cosmic-string evolution.<sup>1</sup> This model has a number of shortcomings, but is a useful and simple way to understand the different processes that take place. The expanding universe is described by a radiation-dominated homogeneous and isotropic cosmology. For simplicity we assume that the metric is spatially flat,

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (1)$$

with a radiation-dominated scale factor

$$a(t) = \sqrt{t/t_0}. \quad (2)$$

The horizon radius in this model is given by

$$l(t) = a(t) \int_0^t a^{-1}(t') dt' = 2t. \quad (3)$$

We will count the production of kinks within a fixed

comoving cube  $x, y, z \in [0, L]$  of comoving volume  $L^3$ . Equivalently, one may identify the opposite faces of the cube to make a three-torus of comoving volume  $L^3$ . (This choice is correct because the string network on large scales is fixed with respect to comoving coordinates; further discussion may be found in Ref. 3.)

At time  $t$ , the physical volume of the spatial section is given by  $V(t) = a^3(t)L^3$  and hence the number of horizon-sized volumes present at time  $t$  is given by

$$N_{\text{horizon}}(t) = V(t)/l^3(t) = L^3(tt_0)^{-3/2}. \quad (4)$$

The reader will note that we have dropped numerical factors of order unity in this calculation. We do this because the one-scale model is already fairly contrived, and cannot be considered as anything other than a rough approximation.

The energy density in infinite strings is<sup>1</sup>

$$\rho_\infty = \nu\mu t/t^3 = \nu\mu t^{-2}, \quad (5)$$

where  $\mu$  is the mass per unit length of the string, and  $\nu$  is the average number of horizon-length segments of infinite string passing through one horizon volume ( $\nu$  is dimensionless). Note that here, and in the rest of this paper, an “infinite” string is one whose length is greater than the horizon length  $l(t)$ . Anything else is called a “loop.”

We will assume that each infinite-string intercommutation cuts one loop of length  $at$  off the infinite-string network. In this one-scale model, loops are formed *only* by being “cut off” the infinite-string network, and their size at the time of formation is a constant fraction  $\alpha$  of the horizon length.

To calculate the number of kinks created per unit time, we need to know how many crossings are taking place. Let  $N_{\text{loop}}(t)$  denote the number of loops formed by intercommutation in the fixed comoving volume  $L^3$  between the time that the string network was formed and time  $t$ . The rate of loop formation  $dN_{\text{loop}}/dt$  is related to the energy density in long strings by

$$\frac{d[V(t)\rho_\infty]}{dt} = -\mu\alpha t \frac{dN_{\text{loop}}}{dt}. \quad (6)$$

(We assume here that the average square velocity of the infinite-string network is  $\frac{1}{2}$ , so that the energy of the string network is conserved. This is a good approximation; a more precise formulation may be found in Appendix A of Ref. 4.) One thus finds a rate of loop formation

$$\frac{dN_{\text{loop}}}{dt} = \frac{v}{2at} N_{\text{horizon}}(t). \quad (7)$$

Note that we are assuming a unit intercommuting probability for the strings.

In order to count the number of kinks on the infinite-string network, we need to know how rapidly kinks are added to and removed from these strings. For each loop formed, four kinks are created. Two of these kinks are added to the infinite-string network, and two are added to the loop which is formed.<sup>5</sup> Then

$$\frac{dn_{\text{created}}}{dt} = 2 \frac{dN_{\text{loop}}}{dt} = \frac{v}{at} N_{\text{horizon}}(t), \quad (8)$$

where  $n_{\text{created}}$  denotes the number of kinks added to the infinite-string network in our fixed comoving volume  $L^3$ .

The infinite-string network loses kinks in two ways. Loops that are cut off the network carry kinks away with them. The remaining kinks are smoothed (damped) away by the stretching of the string due to the expansion of the universe and by the loss of energy due to the emission of gravitational radiation by the kinks. We first consider the case in which *no* smoothing takes place, after which we calculate the effects of smoothing.

If  $K(t)$  denotes the number of kinks per unit (physical) length on the infinite strings, then each loop carries away  $atK(t)$  kinks. Thus,

$$\frac{dn_{\text{removed}}}{dt} = atK(t) \frac{dN_{\text{loop}}}{dt} = \frac{v}{2} K(t) N_{\text{horizon}}(t) \quad (9)$$

gives the number of kinks removed per unit time from the infinite-string network in the fixed comoving volume  $L^3$ .

By subtracting the number of kinks carried off the infinite strings by loop formation from the number added by intercommutings, one obtains the total number of kinks on the infinite-string network. This is related to the linear kink density  $K(t)$  by

$$K(t) = (n_{\text{created}} - n_{\text{removed}})/L_{\infty}, \quad (10)$$

where  $L_{\infty}(t)$  is the total length of infinite string in our fixed comoving volume:

$$L_{\infty}(t) = V(t) \rho_{\infty}/\mu = L^3 v t_0^{-3/2} t^{-1/2}. \quad (11)$$

By differentiating  $L_{\infty}$  times Eq. (10), and using Eqs. (8) and (9), one obtains a differential equation for  $K(t)$ :

$$\frac{d}{dt} [L_{\infty}(t)K(t)] = N_{\text{horizon}}(t) \left[ \frac{v}{at} - \frac{v}{2} K(t) \right]. \quad (12)$$

By substituting in the functional forms of  $L_{\infty}(t)$  and  $N_{\text{horizon}}(t)$  given in Eqs. (4) and (11) one obtains the fol-

lowing differential equation for the kink density  $K(t)$ :

$$\frac{d}{dt} [t^{-1/2}K(t)] = t^{-3/2} \left[ \frac{1}{at} - \frac{K(t)}{2} \right]. \quad (13)$$

This differential equation is easily solved by letting  $\psi(t) = t^{-1/2}K(t)$ .

The general solution to the equation of motion for  $K(t)$  is given by

$$K(t) = (1/at)(qt - 1) \quad (14)$$

(where  $q$  is a free parameter). Imagine that the string network is formed with no kinks at time  $t_{\text{form}}$ , so  $K(t)$  obeys the boundary condition  $K(t_{\text{form}}) = 0$ . The solution is then

$$K(t) = (1/at)(t/t_{\text{form}} - 1). \quad (15)$$

The important point here is that the number of kinks per unit length on the infinite strings is *not* proportional to the inverse of the horizon length; it does not scale. The density  $K(t)$  grows more rapidly than it would for a scaling solution, since a segment of infinite string passing across a given viewer's horizon carries a number of kinks which grows as  $tK(t) \propto (1/a)t/t_{\text{form}}$ . In addition, the loops of length  $at$  being cut off the infinite-string network have a number of kinks which grows in the same way. The nonscaling behavior of the kink density in this case means that the string network does not "forget" its initial conditions [since at late times  $K(t)$  depends upon  $t_{\text{form}}$ ]. This type of behavior was predicted by Bouchet and Bennett,<sup>6</sup> who give a qualitative argument showing that under these conditions the linear density of kinks cannot decrease.

We now consider the effects of kink decay (damping). At least two mechanisms contribute to this: stretching and gravitational radiation. We will see that in a radiation-dominated universe, the latter effect dominates kink decay.

Stretching occurs because in an expanding universe, the angle of a kink is not constant. It is shown in Ref. 7 that the angle of a "typical" kink decays proportionally to a power law:

$$\begin{aligned} \theta(t) &= \theta_k \left[ \frac{a(t)}{a(t_k)} \right]^{2\langle v^2 \rangle - 1} = \theta_k \left[ \frac{t}{t_k} \right]^{(\langle v^2 \rangle - 1/2)} \\ &= \theta_k \left[ \frac{t}{t_k} \right]^{-1/\delta}. \end{aligned} \quad (16)$$

In this formula, the kink is formed with initial angle  $\theta_k$  at time  $t_k$ , and  $\langle v^2 \rangle \approx 0.43$  is the mean velocity squared of the string network in the radiation-dominated era. The constant  $\delta \approx 14$  determines the decay lifetime of a kink due to stretching; a kink formed at time  $t_{\text{birth}}$  disappears at time  $t_{\text{death}} = e^{\delta} t_{\text{birth}}$ .

The effects of kink decay due to emission of gravitational radiation are considered in Refs. 8 and 9, and are similar to the radiative decay of a small loop. Reference

1 shows that a small loop of cosmic string loses energy at a rate  $\dot{E} = -\gamma G\mu^2$ . Thus, a loop of length  $l(t_{\text{birth}})$  will disappear at time  $t_{\text{death}} \approx (1/\gamma G\mu)t_{\text{birth}}$ , where  $\gamma \approx 50$  is a dimensionless parameter. Reference 8 shows that kinks on a cosmic string decay on approximately the same time scale. Taking  $G\mu \approx 10^{-6}$  one finds that  $\delta \approx 10$  due to gravitational radiation provides a faster kink decay mechanism than stretching.

This decay of kinks prevents  $tK(t)$  from growing forever. A simple model permits us to take this effect into account. For convenience, we adopt logarithmic time coordinates  $u, s = \ln(t/t_f)$ , where, as before,  $t_f$  denotes the time of formation of the string network. Let  $n(u, s)ds$  denote the total number of kinks present at (logarithmic) time  $u$  which were formed between (logarithmic) times  $s$  and  $s + ds$  in our fixed comoving volume  $L^3$ . Since  $ds = t^{-1} dt$ , the rate of kink formation (8) per logarithmic time interval is

$$n(s, s) = \nu L^3 \alpha^{-1} t_0^{-3/2} t_f^{-3/2} e^{-3s/2}. \quad (17)$$

The rate of kink loss to loops is given by (9),

$$\frac{\partial}{\partial u} n(u, s) = -\frac{1}{2} n(u, s). \quad (18)$$

We want to know the function  $n(u, s)$  in the upper half quadrant  $0 < s < u$ . The first equation determines  $n(u, s)$  along the diagonal where  $u = s$ , and one can integrate (18) to obtain the solution

$$n(u, s) = \nu L^3 \alpha^{-1} t_0^{-3/2} t_f^{-3/2} e^{-s-u/2}. \quad (19)$$

For simplicity, we assume that kinks formed at time  $t_k$  disappear at time  $t_k e^\delta \approx 2 \times 10^6 t_k$ . The total number of kinks present within the fixed comoving volume  $L^3$  at (logarithmic) time  $u$  is then

$$n(u) = \nu L^3 \alpha^{-1} t_0^{-3/2} t_f^{-3/2} e^{-u/2} \int_{\max(0, u-\delta)}^u e^{-s} ds. \quad (20)$$

The lower limit on the integral is the larger of 0 and  $u - \delta$ , because the first kinks (formed at  $u = 0$ ) begin to decay at  $u = \delta$ . The kink density on infinite strings is now obtained as  $K(u) = n(u)/L_\infty$ . Changing back to physical time, one obtains

$$K(t) = \begin{cases} (1/\alpha t)(t/t_f - 1) & \text{for } t < t_f e^\delta, \\ (1/\alpha t)(e^\delta - 1) & \text{for } t_f e^\delta < t. \end{cases} \quad (21)$$

This function is shown in Fig. 1. For  $t < t_f e^\delta$ , the kink density agrees with the earlier result (15); it rises rapidly until the time at which the first kinks begin to decay. After that time it begins to scale,  $K(t) \propto t^{-1}$ , with the total number of kinks on an infinite-string segment passing through the horizon equal to  $tK(t) = \alpha^{-1} e^\delta$ . For typical values  $\alpha \approx 0.01$  the resulting kink density is much higher than one might expect—about  $10^6$  kinks should be visible on a single string segment passing through the horizon. (Naively, one would only expect  $\sim 1$  kink.)

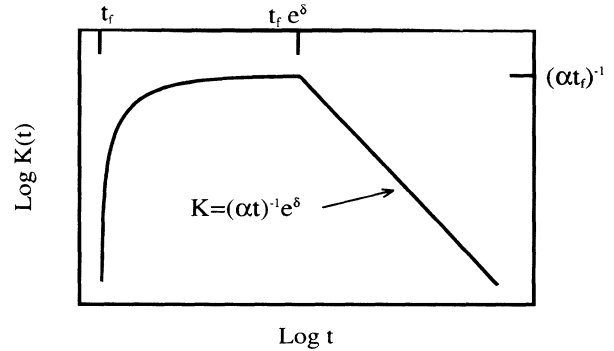


FIG. 1. The linear density of kinks  $K(t)$  as a function of time. The density rises rapidly after the time  $t_f$  when the string network forms. The density reaches scaling  $K(t) \propto t^{-1}$  after the first kinks start to decay at time  $t_f e^\delta$ . At late times, the number of kinks on a string segment stretching across the horizon is about  $\alpha^{-1} e^\delta \approx 2 \times 10^4 \alpha^{-1}$ , which is much larger than one might expect.

The calculation of the kink density is carried out in greater detail in Ref. 4, where the matter-dominated era is also treated. This case differs from the radiation-dominated one. In the radiation-dominated era, the dominant kink decay mechanism is through the emission of gravitational radiation. The faster decay parameter  $\delta \approx 10$  then predicts about  $10^6$  kinks visible on a long string segment. In the matter-dominated era, Ref. 4 shows that the stretching mechanism dominates the kink decay, and that  $\approx 5 \times 10^3$  kinks will be visible on a single string segment passing through the horizon.

In current numerical simulations,<sup>6,10-12</sup> it is clear that small-scale structure is being formed, and this growth of the kink density may suggest the reason why. (Different explanations for this structure formation have also been given, for example, in Refs. 13-16.) In fact, the simulations do not run nearly long enough to reach the scaling behavior in the kink density. A typical radiation-dominated-era simulation runs for  $t_f < t < 25t_f$ , and has  $\alpha < 0.01$ , in which case scaling of the kink density would not occur until  $2 \times 10^4 t_f$  (if the simulation includes the effects of gravitational radiation backreaction) or  $2 \times 10^6 t_f$  (if the effects of backreaction are neglected). Thus, in the simulations, the kink density is predicted to rise rapidly for the duration of the simulation.

The large kink density provides an extremely small length scale  $K^{-1}(t)$  on the string network (i.e.,  $10^{-6}$  of the horizon scale). One can speculate about two possible consequences of this. First, due to the kinky structure, the loops cut off the long strings may be much smaller than naively expected. Because of energy conservation [Eq. (6)], the probability of self-intersection and of loop formation would then increase. This speculation is supported by numerical simulations<sup>7</sup> which show that the rate of loop formation is enhanced by the presence of small-scale structure, and that the loop energy produc-

tion rate is unchanged. If this is so, the true value of  $a$  may be smaller than the value that we use, but the main formulas should still apply. Second, the loops that are cut off the infinite string will have the same high kink density, which may lead to further fragmentation. These types of effects would increase the number of small loops formed, and weaken the existing bounds on  $G\mu$  that come from timing measurements of the binary pulsar.<sup>10,11,17-23</sup> Other effects of kinky strings (which may be very important for galaxy formation) are discussed in Ref. 24.

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<sup>2</sup>D. Garfinkle and T. Vachaspati, Phys. Rev. D **36**, 2229 (1987); **37**, 257 (1988).

<sup>3</sup>An observer whose knowledge is restricted to his own horizon volume will see, for each loop produced,  $atK(t)$  kinks carried away and two added to the reservoir of  $vtK(t)$  kinks on the infinite string visible to him. The naive temptation is to write down the incorrect "equation"

$$\frac{d}{dt}[vtK(t)] = \frac{v}{at} \left( 1 - \frac{atK(t)}{2} \right).$$

The problem here is that the number of kinks within the horizon can increase or decrease even if no string crossings occur, because as the horizon increases in size, more infinite string enters. The correct equation reads

$$\frac{d}{dt}[vtK(t)] = \frac{v}{at} \left( 1 - \frac{atK(t)}{2} \right) + \frac{dn_{\text{entering}}}{dt},$$

where  $n_{\text{entering}} = \frac{3}{2} vtK(t)$  is the number of kinks entering the horizon per unit time on the infinite strings [not the number in the fixed comoving volume, as used in Eq. (9)].

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