Shadowing of Physical Trajectories in Chaotic Dynamics: Containment and Refinement

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For a chaotic system, a noisy trajectory diverges rapidly from the true trajectory with the same initial condition. To understand in what sense the noisy trajectory reflects the true dynamics of the actual system, we developed a rigorous procedure to show that some true trajectories remain close to the noisy one for long times. The procedure involves a combination of containment, which establishes the existence of an uncountable number of true trajectories close to the noisy one, and refinement, which produces a less noisy trajectory. Our procedure is applied to noisy chaotic trajectories of the standard map and the driven pendulum.

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The last decade has witnessed a remarkable pace of development in the understanding of chaotic dynamics. Experiments have made crucial contributions to the development of insights into the behavior of nonlinear systems, and for the calculation of important dynamical quantities. But most experiments, physical and numerical, have noise. In particular, computers have been used to find particle trajectories and chaotic attractors, and to calculate the Lyapunov exponents, the dimension spectra of chaotic attractors and the associated first-order phase transitions, the decay of correlations and diffusion coefficients, and so forth. Computers introduce noise in the system due to truncation errors just as the experimental environment introduces noise in a physical experiment. Moreover, for chaotic processes, neighboring trajectories diverge exponentially from each other. Suppose that truncation error causes errors of order 10^{-8} for processes involving quantities of order 1. If distances between two neighboring trajectories double on the average at each iteration for a given chaotic process, then two trajectories starting 10^{-8} apart will be 1 unit apart in less than 20 iterations.

In Fig. 1, we show a picture of two trajectories for the standard map[1] [see Eq. (1)] that differ only by 10^{-8} on the initial conditions. After 16 iterates their separation, or the error in the dynamical variables, grows to be the same size as the variables themselves, indicating that information about the initial state of the system is, for practical purposes, lost. The numerical investigation of physical models often involves thousands, or even millions, of iterates of a process. Since the relation between the computer-generated trajectory and a true trajectory is no longer clear, the analysis of the physical system is compromised. In view of all this, we are faced with the following central question when interpreting numerical results: For a physical system which exhibits chaos, in what sense does a numerical study reflect the true dynamics of the actual system?

While a noisy trajectory diverges rapidly from the true trajectory with the same initial conditions, there might exist a different true trajectory with slightly different initial conditions which stays near the noisy trajectory for a long time. We have devised a rigorous procedure to prove whether there exists a true trajectory which stays near or *shadows* the noisy trajectory for a long time. When that is the case, the noisy trajectory is an excellent approximation to the true dynamics of the actual chaotic process.

The *shadowing* of the noisy trajectory by a true trajectory was originally discussed for a restricted class of maps,[2] namely, those invertible maps that are *hyperbolic*. This means essentially that each point in the space where the trajectory lies must have a stable direction and an unstable direction, and that under the map infinitesimal displacements in the stable direction decay ex-

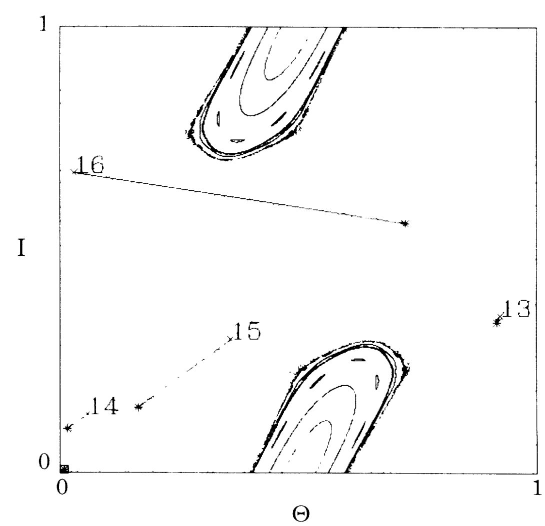

FIG. 1. Two trajectories for the standard map with $K=3.0$. Both trajectories have the same initial conditions $I_0=0.08$ and $\theta_0=0.01$. One of them is iterated using single-precision arithmetic (×) and the other double precision (∗).

ponentially as time goes forward, while infinitesimal displacements in the unstable direction decay exponentially as they are followed backward in time. In addition, it is required that the angles between the stable and unstable directions are uniformly bounded away from zero. If these assumptions are satisfied, it is possible to prove[2] the existence of shadowing for arbitrarily long times. However, typical physical processes are described by systems that are *nonhyperbolic* and their trajectories cannot be shadowed for arbitrarily long times. There exist some results for systems of this type: maps of the interval[3] and two-dimensional dissipative invertible maps.[4]

In this work, we present a general procedure to prove the existence of shadowing for nonhyperbolic chaotic processes. In particular, we establish the existence of shadowing trajectories in two representative Hamiltonian systems, the standard map and the driven pendulum. The standard map, in action-angle variables, has the form

$$\begin{aligned} I_{n+1} &= I_n + (K/2\pi)\sin 2\pi\theta_n \ (\mathrm{mod}\,1)\,, \\ \theta_{n+1} &= \theta_n + I_{n+1} \ (\mathrm{mod}\,1)\,, \end{aligned} \tag{1}$$

and is nonhyperbolic for $K>0$. The standard map is one of the simplest, yet nontrivial, nonlinear Hamiltonian systems. The driven pendulum is described by the differential equation $\ddot{y}+\sin y = f_0\cos t$. Both dynamical systems have been long used as paradigms of Hamiltonian nonlinear systems,[1] and important general properties about Hamiltonian systems were found by studying these systems. It is interesting to note that much of the past work involved the calculation of long numerical trajectories,[1,5] yet shadowing for those numerical trajectories was not proved. We would like to stress that this is the first time the shadowing property has been proved for a chaotic trajectory of a typical (i.e., nonhyperbolic) conservative system.

To explain our work, we start with a few definitions. The term *pseudotrajectory* is used to describe a noisy trajectory.

Definition.— $\{\mathbf{p}_n\}_{n=a}^{b}$ is a $\delta_{\mathbf{f}}$-*pseudotrajectory* for $\mathbf{f}$ if $|\mathbf{p}_{n+1}-\mathbf{f}(\mathbf{p}_n)|<\delta_{\mathbf{f}}$ for $a\le n\le b$, where $\delta_{\mathbf{f}}$ is the noise amplitude and n is an integer. We are interested in the case in which a and b are finite and integers.

Definition.— A *true trajectory* $\{\mathbf{x}_n\}_{n=a}^{b}$ satisfies $\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n)$ for $a\le n\le b$.

Definition of shadowing.— The true trajectory $\{\mathbf{x}_n\}_{n=a}^{b}$ $\delta_{\mathbf{x}}$-*shadows* $\{\mathbf{p}_n\}_{n=a}^{b}$ on $a\le n\le b$ if $|\mathbf{x}_n-\mathbf{p}_n|<\delta_{\mathbf{x}}$ for $a\le n\le b$.

Definition.— The pseudotrajectory $\{\mathbf{p}_n\}_{n=a}^{b}$ has a *glitch* at iterate $n=N<b$ if for some relevant $\delta_{\mathbf{x}}$ there exists a true trajectory that $\delta_{\mathbf{x}}$-shadows $\{\mathbf{p}_n\}_{n=a}^{b}$ for $0\le n\le N$, but no true trajectory that $\delta_{\mathbf{x}}$-shadows it for $0\le n\le N_1$, when $N_1>N$.

To apply our shadowing procedure, we generate a noisy trajectory in a computer. To assure that our results are reproducible and do not depend on specific details of the computer hardware used to generate the pseudotrajectory, we define our own round-off procedure in such a way that it can be duplicated easily. We define a truncation operator $\hat{\mathbf{T}}(\mathbf{x})$ which truncates each coordinate of $\mathbf{x}$ to the B most significant bits. We use $B=48$ ($\sim 10^{-14}$) for the standard map and $B=60$ ($\sim 10^{-18}$) for the driven pendulum. The orbits we analyze are of the form $\mathbf{p}_{n+1}=\hat{\mathbf{T}}[\mathbf{f}(\mathbf{p}_n)]$.

Our objective is to generate a noisy trajectory and then to calculate rigorously *how close* a true trajectory is, and to obtain lower bounds for *how long* a true trajectory stays close. We find that for the standard map (1), with $K=3$ and $\mathbf{p}_0=(0.84,0.54)$, the $\delta_{\mathbf{f}}$ pseudotrajectory $\{\mathbf{p}_n\}_{n=0}^{N}$ where $\mathbf{p}_{n+1}=\hat{\mathbf{T}}[\mathbf{f}(\mathbf{p}_n)]$, is $\delta_{\mathbf{x}}$-shadowed by a true trajectory with $\delta_{\mathbf{x}}=10^{-8}$ for $N=10^7$ iterates. The long shadowing time is striking when compared to the great rate at which nearby orbits diverge from each other; in this case roughly a factor of 3 on each iteration. We have chosen the initial conditions $\mathbf{p}_0=(0.84,0.54)$ to appear to be typical. We have obtained comparable results for a variety of other parameter values (see Table I).

For the driven pendulum $\ddot{y}+\sin y=f_0\cos t$, with $f_0=2.4$ and initial condition $\mathbf{p}_0=(y_0,\dot{y}_0)=(0,0)$ at time $t=0$, a $\delta_{\mathbf{f}}$-pseudotrajectory was created with a maximum truncation error of $\delta_{\mathbf{f}}=10^{-18}$. The pseudotrajectory appears to be chaotic for this parameter and these initial conditions. Our techniques allow us to prove the existence of a true shadowing trajectory within $\delta_{\mathbf{x}}=10^{-8}$ for time t ranging between 0 and $10^4\pi$. Again there are similar results for other values of $\mathbf{p}_0$ and f_0. The pendulum calculations were done using a seventh-order Taylor-series integration method with an explicit truncation-error formula. The flow was calculated at times h units apart to generate a discrete trajectory $\{\mathbf{p}_n\}_{n=0}^{N}$. For the results reported here, we used $h=\pi/1000$.

Our technique to prove shadowing for a nonhyperbolic system involves a combination of (i) containment of a true trajectory, and (ii) refinement of the noisy trajectory.

TABLE I. Shadowing distance $\delta_{\mathbf{x}}$ for various values of the parameter K, where $\mathbf{p}_0=(0.84,0.54)$ and $N=10^6$ in all cases.

K	$\delta_{\mathbf{x}}$
1.0	2.9×10^{-7}
1.05	5.4×10^{-8}
2.25	6.1×10^{-10}
2.5	2.6×10^{-10}
3.0	2.9×10^{-9}
4.0	1.7×10^{-10}
5.0	3.5×10^{-10}
7.5	9.5×10^{-11}

The *containment of a true trajectory* requires the construction of a sequence of small parallelograms, $\{M_n\}_{n=0}^N$. The parallelograms must be constructed so that the image $\mathbf{f}(M_n)$ lies across M_{n+1} as shown in Fig. 2. There is also an orientation requirement. Two parallel sides of each M_n are designated as *expanding* sides, and the images of the expanding sides of M_n must intersect the two contracting sides of M_{n+1} but cannot intersect the expanding sides of M_{n+1}. In practice it is necessary to have an upper bound on the sizes of the second partial derivatives of $\mathbf{f}$ in order to guarantee that the images of the expanding sides of M_n do not bend so much that they touch the expanding sides of M_{n+1}. When we have such parallelograms, we say it is a *containing sequence* of parallelograms $\{M_n\}_{n=0}^N$.

We now argue that there must be a true trajectory $\{\mathbf{x}_n\}_{n=0}^N$ contained in $\{M_n\}_{n=0}^N$ with $\mathbf{x}_n$ contained in M_n for $0 \le n \le N$. Let γ_0 be a curve lying wholly in M_0 as shown in Fig. 2 running from one of the contracting sides of M_0 to the other. Then $\mathbf{f}(\gamma_0)$ contains a curve γ_1 that lies wholly in M_1 and runs from one contracting side of M_1 to the other. In fact, there exist curves γ_{n+1} contained in $\mathbf{f}(\gamma_n)$ that lie wholly in M_{n+1}. Select any point on the final curve γ_N and call it $\mathbf{x}_N$. Then $\mathbf{x}_{N-1}$, defined to be $\mathbf{f}^{-1}(\mathbf{x}_N)$, lies on γ_{N-1} and so lies in M_{N-1}. Continuing backwards, $\mathbf{x}_n$ is defined to be $\mathbf{f}^{-1}(\mathbf{x}_{n+1})$ for $0 \le n \le N$, giving then a true trajectory $\{\mathbf{x}_n\}_{n=0}^N$ contained in $\{M_n\}_{n=0}^N$. Hence the sequence of parallelograms in fact contains a true trajectory.

To find the shadowing distance, we compute the distance of the nth point $\mathbf{p}_n$ of the original pseudotrajectory to the furthest point of the nth parallelogram M_n, and then take the maximum of these distances along the whole trajectory. Note that the term "parallelogram" connotes a two-dimensional figure, and the examples of this paper fit within this context. However, the preceding argument and other techniques we present go over without essential change to phase spaces of higher dimensions.

The expanding and contracting sides of a parallelogram M_n are parallel to the local unstable $\mathbf{u}_n$ and stable $\mathbf{s}_n$ unit vectors at a point $\mathbf{p}_n$, respectively. The unit vectors follow the linearized map, i.e.,

$$\mathbf{u}_{n+1} = L_n \mathbf{u}_n / |L_n \mathbf{u}_n| \tag{2a}$$

and

$$\mathbf{s}_{n+1} = L_n \mathbf{s}_n / |L_n \mathbf{s}_n| \,, \tag{2b}$$

where L_n is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{p}_n$. For an arbitrary initial unit vector $\mathbf{u}_0$, Eq. (2a) gives $\mathbf{u}_n$ nearly aligned with the unstable direction at $\mathbf{p}_n$ after just a few iterates, while, for an arbitrary initial unit vector $\mathbf{s}_N$, Eq. (2b) is iterated backwards and gives $\mathbf{s}_n$ nearly aligned with the stable direction at $\mathbf{p}_n$ after just a few iterates.

For a chaotic trajectory in a nonhyperbolic system, containment will not continue forever. Containment breaks down when no parallelogram M_{n+1} can be found to make a "plus" sign with $\mathbf{f}(M_n)$. These glitches are rare, but do occur when an angle of the parallelogram becomes nearly zero, so that the parallelogram effectively loses a dimension. This occurs when the angle between the stable and unstable directions becomes small relative to the noise level $\delta_\mathbf{f}$ of the pseudotrajectory. It follows that the lower the noise level, the longer the shadowing trajectory will be. We make a quantitative conjecture of this relationship at the closing of the paper.

The *refinement* technique is essentially a method of noise reduction which is used to enhance the success of the containment procedure. (It can also be used for noise reduction of experimental data.) Given the pseudotrajectory $\{\mathbf{p}_n\}_{n=0}^N$, the refinement process produces a less noisy pseudotrajectory $\{\tilde{\mathbf{p}}_n\}_{n=0}^N$ which remains uniformly near $\{\mathbf{p}_n\}_{n=0}^N$ and whose points serve as the centers of the parallelograms $\{M_n\}_{n=0}^N$ in the containment procedure.

Let $\boldsymbol{\pi}_{n+1}$ represent the one-step noise

$$\boldsymbol{\pi}_{n+1} = \mathbf{p}_{n+1} - \mathbf{f}(\mathbf{p}_n) \,, \tag{3}$$

where it is assumed that $|\boldsymbol{\pi}_{n+1}| < \delta_\mathbf{f}$. (For our trajectories, we use $\tilde{\mathbf{T}}[\mathbf{f}(\mathbf{p}_n)]$ instead of $\mathbf{f}(\mathbf{p}_n)$.) The refined orbit $\{\tilde{\mathbf{p}}_n\}$ is constructed by setting

$$\tilde{\mathbf{p}}_n = \mathbf{p}_n + \boldsymbol{\Phi}_n \,. \tag{4}$$

The equation satisfied by $\boldsymbol{\Phi}_n$, using Eqs. (3) and (4), is then

$$\boldsymbol{\Phi}_{n+1} = \mathbf{f}(\tilde{\mathbf{p}}_n) - \boldsymbol{\pi}_{n+1} - \mathbf{f}(\mathbf{p}_n) \,, \tag{5}$$

where $\tilde{\mathbf{p}}_{n+1} = \mathbf{f}(\tilde{\mathbf{p}}_n)$. Requiring $\boldsymbol{\Phi}_n$ to be small, we can expand $\mathbf{f}(\tilde{\mathbf{p}}_n)$ about $\mathbf{p}_n$ in Taylor series to get $\mathbf{f}(\tilde{\mathbf{p}}_n) \simeq \mathbf{f}(\mathbf{p}_n) + L_n \boldsymbol{\Phi}_n$. Hence, Eq. (5) becomes

$$\boldsymbol{\Phi}_{n+1} = L_n \boldsymbol{\Phi}_n - \boldsymbol{\pi}_{n+1} \,. \tag{6}$$

The objective is to find $\{\boldsymbol{\Phi}_n\}_{n=0}^N$, and hence $\{\tilde{\mathbf{p}}\}_{n=0}^N$ by Eq. (4), in the coordinates $\{\mathbf{u}_n\}_{n=0}^N$ and $\{\mathbf{s}_n\}_{n=0}^N$. For that, we represent $\boldsymbol{\Phi}_n$ as $\boldsymbol{\Phi}_n = \alpha_n \mathbf{u}_n + \beta_n \mathbf{s}_n$ and $\boldsymbol{\pi}_n$ as $\boldsymbol{\pi}_n = \eta_n \mathbf{u}_n + \zeta_n \mathbf{s}_n$, respectively. Observe that $\mathbf{f}(\mathbf{p}_n)$ in Eq. (3) can be well approximated by $\tilde{\mathbf{f}}(\mathbf{p}_n)$, the noisy image of $\mathbf{p}_n$. Thus, given $\{\mathbf{p}_n\}_{n=0}^N$, $\{\eta_n\}_{n=0}^N$ and $\{\zeta_n\}_{n=0}^N$ can be

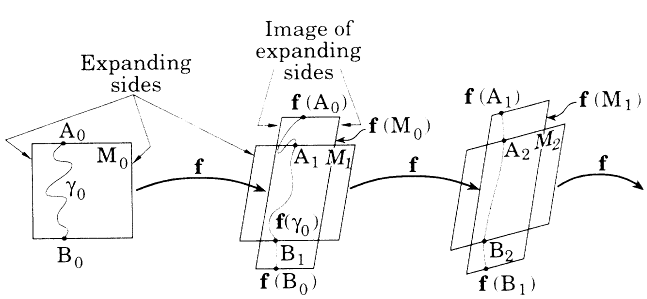

FIG. 2. Containment of a true trajectory. γ_1 is the piece of $f(\gamma_0)$ that runs from A_1 to B_1.

calculated directly from Eq. (3). To find $\{\alpha_n\}_{n=0}^N$ and $\{\beta_n\}_{n=0}^N$ in terms of $\{\eta_n\}_{n=0}^N$ and $\{\zeta_n\}_{n=0}^N$, rewrite Eq. (6) as

$$\mathbf{\Phi}_{n+1}=L_n(\alpha_n\mathbf{u}_n+\beta_n\mathbf{s}_n)-(\eta_{n+1}\mathbf{u}_{n+1}+\zeta_{n+1}\mathbf{s}_{n+1}), \quad (7)$$

where $\mathbf{\Phi}_{n+1}=\alpha_{n+1}\mathbf{u}_{n+1}+\beta_{n+1}\mathbf{s}_{n+1}$. The unit vectors follow the linearized map. The substitution of Eqs. (2) in (7) yields recursive relations for $\{\alpha_n\}_{n=0}^N$ and $\{\beta_n\}_{n=0}^N$:

$$\alpha_{n+1}=|L_n\mathbf{u}_n|\alpha_n-\eta_{n+1},$$
$$\beta_{n+1}=|L_n\mathbf{s}_n|\beta_n-\zeta_{n+1}. \quad (8)$$

Equations (8) are made computationally stable by calculating the coefficients α_n in the unstable direction $\mathbf{u}_n$ by starting at the end point $n=N$, and the coefficients β_n in the stable direction $\mathbf{s}_n$ by starting at the initial point $n=0$:

$$\alpha_n=(\alpha_{n+1}+\eta_{n+1})/|L_n\mathbf{u}_n|, \quad \alpha_N=0, \quad (9a)$$

and

$$\beta_{n+1}=\beta_n|L_n\mathbf{s}_n|-\zeta_{n+1}, \quad \beta_0=0. \quad (9b)$$

The refinement computations are carried out using higher accuracy than the noise level of $\{\mathbf{p}_n\}_{n=0}^N$. We used 96-bit ($\sim10^{-29}$) arithmetic for the refinement step. The pseudotrajectory $\{\tilde{\mathbf{p}}_n\}_{n=0}^N$ is less noisy than the original $\{\mathbf{p}_n\}_{n=0}^N$. In fact, when the refinement step is iterated, the procedure is superconvergent: The number of significant digits typically doubles on each iteration of the process. Of course, at a glitch no decrease in the noise may be possible.

In conclusion, we have a rigorous procedure to shadow noisy trajectories by true trajectories for nonhyperbolic systems, which are the systems typically found in nonlinear dynamics. To obtain parallelograms and optimal bounds on the shadowing distance, we combine the procedure for the containment of a true trajectory with the procedure for the refinement of the noisy trajectory. Table I shows shadowing results for various values of the parameter of the standard map. Observe that we can shadow chaotic processes down to the critical value of the parameter $K\simeq0.97$ when the last Kolmogorov-Arnol'd-Moser surface is broken and when the islands of stability are large. We have applied our procedure to other chaotic trajectories of the standard map and pendulum as well as to other dynamical systems.[4]

In order to indicate the relative magnitudes of the quantities we expect to find if we consider a different physical model, we present the following *conjecture:* For a typical two-dimensional Hamiltonian map yielding chaotic trajectories with a small noise amplitude $\delta_\mathrm{f}>0$, we expect to find $\delta_\mathbf{x}\le\sqrt{\delta_\mathrm{f}}$ for a trajectory of length $N\simeq1/\sqrt{\delta_\mathrm{f}}$.

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