Lyapunov Exponents from Observed Time Series

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We examine the question of accurately determining Lyapunov exponents for a time series. We find that it is advantageous to use local mappings with higher-order Taylor series, rather than linear maps as done earlier. We demonstrate this procedure for the Ikeda map and the Lorenz system. We present methods for identifying spurious exponents by analyzing data-set singularities and by determining the Lyapunov direction vectors. The behavior of spurious exponents in the presence of noise is also investigated, and found to be different from that of the true exponents.

Determining the Lyapunov exponents of a nonlinear system from measurements of a time series is an important challenge for any analysis of the dynamics. Positive exponents are generally regarded as equivalent to the presence of real dynamical chaos, and the Lyapunov exponents are classifiers of the dynamics since they are characteristic of the attractor and independent of any given orbit or initial condition.[1,2] If the governing equations are known, then there are reliable methods for determining all of the exponents. If one only has a time series, then the problem becomes much more difficult. There are several reported efforts to provide algorithms for the determination of the Lyapunov exponents from observations alone.[3-6] Our own experience[7] with these algorithms is that they are reliable only for the largest exponent and not for the others. The importance of Lyapunov exponents in the study of physical systems has led us to provide an improvement on these previous efforts and to address several other questions of importance in the determination of Lyapunov experiments from data. In this Letter we report on the basic outline of our methods. We leave details and numerous other examples to our longer paper.[8]

Earlier work and ours assume that a scalar time series $x(t_0 + n\tau) = x(n)$, $n = 1, 2, \ldots, N_D$, has been observed.

From this the phase space of the system has been reconstructed by the familiar time-delay method[8-12] to produce data vectors in d dimensions:

$$\mathbf{y}(n) = [x(n), x(n+T), \ldots, x(n+T(d-1))] .$$

In discretized time one takes the dynamics to be a map of R^d to itself which evolves the vectors $\mathbf{y}(n)$: $\mathbf{y}(n + T_2) = \mathbf{F}(\mathbf{y}(n))$, where the time delay T_2 is independent of T. The product of the Jacobians of this map $\mathbf{DF}(\mathbf{y}) = \partial \mathbf{F}(\mathbf{y})/\partial \mathbf{y}$ evaluated along an orbit contains the information required for the Lyapunov exponents.[1,2]

The first step in the analysis is to find the neighboring points of a given point in the data set. Our choice of neighbors is limited by the finite size of the data set, by stochastic noise, and most importantly by the fractal nature of the attractor. These limitations are the main source of difficulties in the analysis. Finding legitimate neighbors of a given point is one of the most critical tasks in obtaining accurate results. For this reason it is often advisable to maintain two different dimensions for converting the scalar data set into time-delay vectors. The first is a "local dimension" d which is equal to the number of Lyapunov exponents that the calculation will produce and is the dimension of the Jacobian matrices. The second is a "global dimension" d_G which is used in

the process of identifying neighbors. d_G can be made larger than d in order to insure that we do not have any false neighbors entering the calculation as might be the case if the attractor is folded in such a way that it crosses itself. One way of choosing values for d and d_G is to start by computing a rough estimate of the fractal dimension d_A of the attractor. If an object of dimension d_A is mapped in a very general way into a space of dimension d_G it can typically have self-intersections of dimension $2d_A - d_G$. Thus we would like to make d_G larger than $2d_A$. It is not advisable to make d much greater than d_A, since locally the attractor has very little extension into these additional directions. A good first choice is to try making d the next integer greater than or equal to the fractal dimension d_A. This choice will always give at least one negative exponent for a chaotic system. One is then free to further increase d and see what effect this has on the results.

Earlier work attempts to find the required Jacobians by making local *linear* maps of neighborhoods near the orbit $y(n)$ to neighborhoods at a subsequent time. We depart from these earlier works by making local polynomial maps, allowing for a more accurate determination of $DF(y)$. While there have been previous uses of higher-order mappings in studies of dynamical systems (see, e.g., Refs. 13 and 14) this is, we believe, the first application of this approach to the calculation of the full Lyapunov spectrum. The vector from the rth neighbor to an orbit point $y(n)$ is denoted by z^r. Using a least-squares fit to the data we obtain a polynomial map from this vector at time 0 to the same vector on time step T_2 later: $z^r(n;0) \rightarrow z^r(n;T_2)$, including all terms up to some specified order N_{Tay}. We have examined the effects of retaining terms up to fifth order in $z^r(n;0)$. We use at least twice the number of neighbors as parameters to be determined in order to insure reliable results. We then proceed to calculate the Lyapunov exponents using the QR decomposition technique discussed by Eckmann *et al.*[3,15,16] As we show with our methods, one can determine the positive, zero, and often one or more negative exponents.

In this Letter we report on results from the Ikeda[17] map of the complex plane to itself,

$$z(n+1) = p + Bz(n)\exp\{i\kappa - i\alpha/[1 + |z(n)|^2]\}, \quad (1)$$

where $p = 1.0$, $B = 0.9$, $\kappa = 0.4$, and $\alpha = 6.0$. For these parameters we calculate (using the map) that λ_1 and λ_2 are 0.503 and -0.719, respectively. We also study the Lorenz system of three ordinary differential equations:[18]

$$\frac{dx_1}{dt}(t) = \sigma[x_2(t) - x_1(t)],$$

$$\frac{dx_2}{dt}(t) = -x_1(t)x_3(t) + rx_1(t) - x_2(t), \quad (2)$$

$$\frac{dx_3}{dt}(t) = x_1(t)x_2(t) - bx_3(t),$$

where we take $\sigma = 16$, $b = 4$, and $r = 45.92$. For these parameters the accepted values for the Lyapunov exponents are 1.50, 0.00, and -22.5, respectively. The large negative exponent makes this system a particularly challenging test for our time-series method, and also requires the use of very accurate data, as will be shown.

The difficulty in determining the negative exponents from a time series comes primarily from the fact that the attractor is often very "thin" at many locations in the directions associated with certain negative exponents. Even when there is a reasonably large and accurate data set, this will often make curvature effects within a given neighborhood become significant. A linear analysis becomes totally inaccurate when the displacement due to local data-set curvature is comparable to the thickness of the data set. Going to a higher-order approximation of the mapping can correct this.

The Ikeda map is an excellent example of a system for which the use of separate local and global dimensions is important. Examination of the two-dimensional time-delay representation Fig. 1 shows clearly the self-intersection effect which was discussed previously. Having determined the fractal dimension d_A to be about 1.8 we would choose d_G to be at least 3 and preferably 4, and an appropriate value for d is 2. Using the incorrect values $d_G = d = 2$ in a third-order calculation we obtain $\lambda_1 = 0.565$ and $\lambda_2 = -0.426$, while if we use $d_G = 4$ and $d = 2$ we obtain $\lambda_1 = 0.512$ and $\lambda_2 = -0.736$ which are much closer to the correct values (0.503 and -0.719). If we keep d equal to d_G but increase both of their values to 3, the overlap problem is reduced but we now obtain 3 exponents (0.554, -0.262, and -0.821) and so we are faced with the problem of deciding which, if any, are valid exponents.

We move now to the Lorenz system. In the case of data from the Lorenz equations we have used two slightly different settings for the evolution time lag. We display the results of both our calculations in Table I.
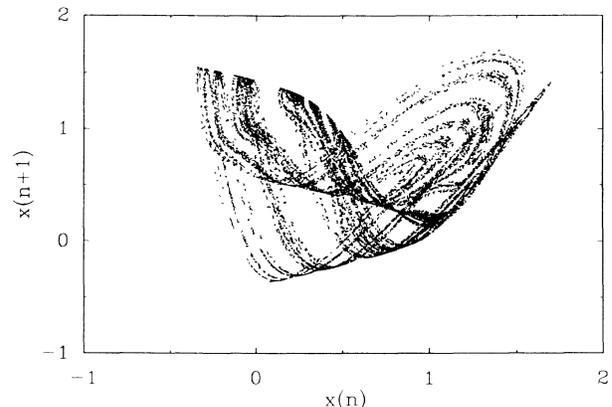
FIG. 1. Reconstructed phase portrait of the Ikeda map in $d_G = 2$. The lack of a one-to-one projection of the attractor onto the $[x(n), x(n+1)]$ plane is clear to the eye.

TABLE I. In the top part of the table we display the Lyapunov exponents for the Lorenz system computed from 50 000 data points evaluated with a sampling time $\tau = T_2 = 0.02$ and a time delay $T = 5\tau = 0.1$. The data have five digits of accuracy and are analyzed using $d = d_G = 3$ for varying orders N_{Tay} of the mapping. In the bottom part we display the Lyapunov exponents for the Lorenz system computed from 20 000 data points evaluated with a sampling time $\tau = 0.05$ and a time delay $T = T_2 = 2\tau = 0.1$. The data have 9+ digits of accuracy and are analyzed using $d = 3$ and $d_G = 7$ for mapping orders 1-5.

N_{Tay}	λ_1	λ_2	λ_3
1	1.4504	−0.005 712 3	−13.999
2	1.5027	−0.046 041	−19.448
3	1.5121	0.006 964 1	−22.925
4	1.5561	0.032 219	−23.465
1	1.549	−0.094 70	−14.31
2	1.519	−0.026 47	−20.26
3	1.505	−0.005 695	−22.59
4	1.502	−0.002 847	−22.63
5	1.502	−0.000 387	−22.40

As the reader will observe, the negative exponent is very difficult to obtain, and here we see it dramatically "snapping into place" as we increase the order of the calculation to 3 and above. Also note the improvement in accuracy of the zero exponent in the lower part of Table I. The greater accuracy of the lower part of Table I is due primarily to the higher accuracy of the data set.

In Table II, we analyze the Lorenz equations with a local dimension $d = 4$, which we know must generate at least one spurious exponent. When using a second-order fitting to our local map, the results are poor for all of the Lyapunov exponents. Increasing the polynomial fitting to third order we find that the last three exponents are very close to the true exponents, while the first is 10 times larger than the true value of the largest exponent.

In addition to obtaining the Lyapunov exponents, one can also obtain the direction vectors $\mathbf{L}_i$ associated with these exponents. The $\mathbf{L}_i$ are defined by the requirement that a small displacement along any one of these directions followed forward or backward in time will expand or contract on average at the rate given by the corre-

sponding exponent. Although they are different at every location on the attractor, their calculation requires a knowledge of the orbit far into the past and future for a given point on the attractor. The details of their calculation are to be found in our longer paper.[8] They should be examined to see if two or more of them are nearly collinear. This can occur if a poor choice was made for the delay time (probably too small) or if nonlinear effects are generating a spurious exponent. The spurious nature of λ_1 in our Lorenz-system calculations can be rapidly identified by examining the local data thickness Th_1 in the $\mathbf{L}_1$ direction, which is over 5 orders of magnitude smaller than the thickness Th_2 for $\mathbf{L}_2$. A valid positive exponent should not exhibit any significant "thinness," and it is preferable for all exponents to have thickness levels above the intrinsic noise level of the data set. The thickness Th_i is essentially the rms displacement of the data points within a local neighborhood in the $\mathbf{L}_i$ direction, with corrections for data-set curvature; more details can be found elsewhere.[8]

Although we have shown that it is possible to include singular directions in the calculation and later identify the questionable exponents, the presence of relatively small amounts of noise makes this more difficult. This is illustrated in Fig. 2 for the Lorenz system. We have added Gaussian white noise to the data points with the indicated standard deviation. In Fig. 2(a) we have used $d = 3$, while in 2(b) we used $d = 4$, which gives one spurious exponent. In both cases we used a third-order expansion for the local mappings. The spurious exponent in 2(b) drops rapidly as the added noise is increased, going from +19 down to −6. This behavior is in fact another way of identifying a spurious exponent in extremely accurate data. However, the absence of such a drop does not guarantee that spurious exponents do not exist.

We conclude by noting that a simple extension to higher order of earlier methods for determining the Lyapunov spectrum for a dynamical system from observations alone works strikingly well when tested on familiar systems such as the Ikeda map and Lorenz attractor. We have also suggested and tested on these examples ways to determine which of the exponents are valid and which are spurious. Finally, we explored the effect of numerical accuracy and external noise on the determina-

TABLE II. Lyapunov exponents and thicknesses (Th) of the attractor along the corresponding Lyapunov direction vectors for the Lorenz system, through order $N_{\text{Tay}} = 4$. The calculation was done with 20 000 data points using $d = 4$ and $d_G = 7$ so that there is one spurious exponent. For $N_{\text{Tay}} = 2$ and above, the spurious exponent separates from the true ones and can be identified by its extremely small thickness value.

N_{Tay}	λ_1	Th_1	λ_2	Th_2	λ_3	Th_3	λ_4	Th_4
1	1.936	0.467	0.802	1.083	−1.137	1.111	−13.44	0.162
2	4.36	0.002 56	1.401	0.412	−0.656	0.454	−20.57	0.001 16
3	18.04	1.89×10^{-7}	1.502	0.129	−0.0005	0.091	−22.77	2.7×10^{-4}
4	26.96	2.5×10^{-10}	1.503	0.066	−0.0048	0.066	−22.55	5.6×10^{-5}
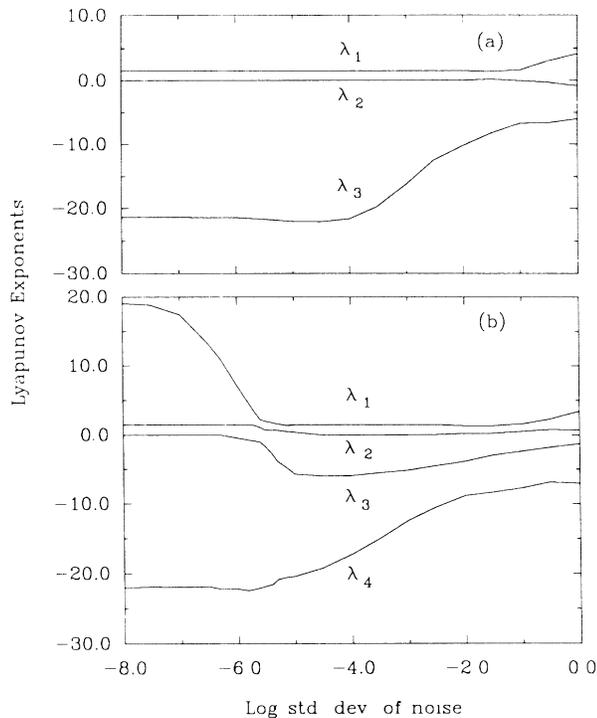
FIG. 2. The effect of external noise on the determination of Lyapunov exponents for the Lorenz system. (a) Here local dimension is $d=3$. (b) Here $d=4$. The spurious exponent wanders from about $+19$ to nearly -6 as the noise level is varied. Note that the exponents do not cross each other but prefer to switch roles as they become close. In the $d=3$ case the correct exponents are more robust against the addition of noise.

tion of these exponents. We found the calculation to be quite sensitive to noise, which reflects the fact that fluctuations in phase-space points make determination of the required local Jacobians quite sensitive. Further examples and extensive details on the methods outlined here will be found in our longer paper.[8] Also, the algorithms used here are available on request from the authors.

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