Conductance Distribution at the Mobility Edge

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It is shown that, for a system at the mobility edge, the conductance distribution $P_L(g)$ approaches an entirely universal function $P^*(g)$ when the size L of the system increases. The function $P^*(g)$ depends on dimensionality d alone and does not depend on any details of the model. In obtaining $P^*(g)$ use was made of the conductance cumulants calculated by Altshuler, Kravtsov, and Lerner [Phys. Lett. A **134**, 488 (1989)]. The calculations are limited to spatial dimension $d=2+\epsilon$, where ϵ is a small number.

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In quantum transport one is usually interested in the conductance $G(L)$ of a disordered electronic sample, which is assumed to be a d-dimensional hypercube of size L. At sufficiently low temperatures (the mesoscopic regime) the sample should be viewed as one coherent unit (a huge molecule) and its conductance is sensitive to the precise impurity arrangement in the sample. Even in a good metal there are anomalously large conductance fluctuations from sample to sample.[1] Under increase of disorder the system approaches the Anderson transition point and then crosses over into the insulating regime. At the transition, and in the insulator, fluctuations in the conductance become very large and it therefore becomes necessary to study the full distribution of conductances $P_L(G)$, for an ensemble of macroscopically identical samples of size L.[2-8] In particular, one would like to know whether $P_L(G)$ approaches a universal limiting function when the size L of the system increases. It is clear that such a limiting distribution can be achieved, if at all, only in the $L\rightarrow\infty$ limit (and for the appropriate variable). It is important, however, to realize that already for finite L the distribution $P_L(G)$ might become approximately universal and independent, to a large extent, of the microscopic details specifying the system or the model.

The significance of limiting distributions and their relation to scaling[9] was discussed in detail in Ref. 5. It was argued there that universal limiting distributions do indeed exist and that, in the case of weak disorder, they can be specified by just one parameter. This means that the limiting distribution is completely determined by the choice of a single parameter which is a measure of the local disorder in the system. If this parameter is tuned to some critical value, corresponding to the mobility edge, the distribution $P_L(G)$ evolves towards an absolutely universal function $\lim_{L\rightarrow\infty} P_L(G)=P^*(G)$ which depends on the dimensionality alone. (One should keep in mind that, for locally weak disorder, the Anderson transition can occur only for $d=2+\epsilon$ where ϵ is a small number.) The calculations in Ref. 5 were done for a toy model (derived by a Migdal-Kadanoff-type approach) which, although displaying a metal-insulator transition, was by no means a realistic model of a genuine d-dimensional electronic system. It is the purpose of this Letter to derive the limiting distribution $P^*(G)$, for a system at the mobility edge, using the standard model[1,4,6] of noninteracting electrons moving in the presence of a d-dimensional random potential. In what follows we shall make use of the conductance cumulants, $C_n(L)$, calculated by Altshuler, Kravtsov, and Lerner[6] [$C_n(L)$ denotes the nth cumulant of the conductance distribution $P_L(g)$ where $g\equiv\pi^2\hbar G/e^2$ is the dimensional conductance]. It was shown in Ref. 6 that, for $d=2+\epsilon$ and at the mobility edge,

$$C_n(L)=\begin{cases}\epsilon^{n-2}, & n\lesssim n_0\simeq 1/\epsilon,\\ (L/l)^{\epsilon n^2-2n}, & n\gtrsim n_0,\end{cases} \tag{1}$$

where l is the elastic mean free path and n_0 is a large integer of order $1/\epsilon$. According to Eq. (1) the low-order cumulants ($n<n_0$) scale, under change of L, towards universal numbers ϵ^{n-2}. However, the high-order cumulants ($n>n_0$) increase with L indefinitely and do depend (through the mean free path) on the model. It seems, at first sight, that such a behavior of the high-order cumulants precludes having an absolutely universal (i.e., independent of l) distribution $P^*(g)$, in the limit of $L\rightarrow\infty$. Below we show that this is not so and that there is no contradiction between Eq. (1) and the existence of a universal limit $\lim_{L\rightarrow\infty} P_L(g)\equiv P^*(g)$.

Before proceeding further, let us return to the toy model of Ref. 5. The great advantage of this model is that it enables one to work directly with the full distribution rather than only with its moments (or cumulants). One can therefore avoid some ambiguities and guesses which will be involved in the reconstruction of the distribution $P_L(g)$ for the "real" problem from its cumulants [Eq. (1)]. For the toy model it has been possible to derive a differential equation[5] for the resistance distribution $W_L(\rho)$ where $\rho=1/g$ and L is measured in some microscopic units. This equation contains as a parameter the first moment of $W_L(\rho)$, i.e., the averaged resistance $\bar{\rho}_L$. The value $\bar{\rho}_L=\rho_c=\epsilon+O(\epsilon^2)$ corresponds to the mobility edge, and the equation for $W_L(\rho)$ is then

$$\frac{\partial W}{\partial \zeta}=\frac{\partial}{\partial \rho}\left[\epsilon(\rho^2+\rho)\frac{\partial W}{\partial \rho}+(1+\epsilon)\rho W\right], \qquad (2)$$

where $\zeta \equiv \ln L$. Equation (2) is a Fokker-Planck equation and one can prove by standard means[10] that any initial distribution $W_0(\rho)$ evolves towards the same limiting distribution $\lim_{L\to\infty} W_L(\rho)\equiv W^*(\rho)$. This limiting distribution is obtained by setting the right-hand side of Eq. (2) to zero and solving the resulting ordinary differential equation. This gives[5,11]

$$W^*(\rho)=(1/\epsilon)(1+\rho)^{-1/\epsilon-1}. \qquad (3)$$

Let us now look at the moments $\mu_n(L)\equiv\int d\rho\, \rho^n W_L(\rho)$ of the distribution $W_L(\rho)$. Multiplying (2) by ρ^n and integrating over ρ, one obtains the following recursion equations for μ_n:

$$\partial\mu_n/\partial\zeta=(\epsilon n^2-n)\mu_n+\epsilon n^2\mu_{n-1}. \qquad (4)$$

It follows from these equations that the low-order moments ($n<1/\epsilon$) approach, in the limit of large L, some constant values μ_n^* related to each other by[5] ($\mu_0^*=1$ by normalization)

$$\mu_n^*=[\epsilon n/(1-\epsilon n)]\mu_{n-1}^* \quad (n<1/\epsilon). \qquad (5)$$

On the other hand, high-order moments ($n>1/\epsilon$) keep increasing with L as

$$\mu_n \simeq A_n e^{(\epsilon n^2-n)\zeta}=A_n L^{\epsilon n^2-n} \quad (n>1/\epsilon) \qquad (6)$$

and are sensitive to the initial distribution $W_0(\rho)$ [the nonuniversal coefficients A_n in Eq. (6)]. One concludes, thus, that nonuniversal, and growing with L, moments [Eq. (6)] are perfectly compatible with the existence of an entirely universal limiting distribution [Eq. (3)]. As explained in Ref. 5, such a behavior of the high-order moments indicate that these moments are completely dominated by the tails of the distribution, i.e., by rare, statistically insignificant events.

Let us now imagine that the equation for $W_L(\rho)$ [Eq. (2)] is not known and that all one knows are the moments of the distribution, for large L [Eqs. (5) and (6)]. Can one reconstruct the distribution $W_L(\rho)$ from these moments? The answer is, perhaps a bit surprisingly, negative. The point is that for a fixed L (larger than l) the high-order moments grow very fast with their number n (faster than $n!$), so that the problem of moments (i.e., determining the distribution from its moments) has no unique solution.[12] It is possible, nevertheless, to go back from the moments [Eqs. (5) and (6)] to the distribution $W_L(\rho)$ by making certain plausible assumptions. Here is a possible route: First, derive from Eqs. (5) and (6) the underlying Eq. (4) describing the evolution of the moments μ_n with L, or ζ [in doing so one assumes that these underlying equations contain only first-order derivatives with respect to ζ; this assumption is equivalent to the statement that the distribution $W_L(\rho)$ at some arbitrary scale L is completely determined by its shape $W_L(\rho)$ at some initial scale L_0]. Second, multiply the obtained equations [Eq. (4)] by $(-\lambda)^n/n!$ and sum over n, which leads to the following equations for the quantity $\Gamma(\lambda)=\sum_n(-\lambda)^n\mu_n/n!$ (the generating function for the moments μ_n):

$$\frac{\partial\Gamma}{\partial\zeta}=\epsilon\lambda^2\frac{\partial^2\Gamma}{\partial\lambda^2}-\lambda(1-\epsilon+\epsilon\lambda)\frac{\partial\Gamma}{\partial\lambda}-\epsilon\lambda T. \qquad (7)$$

One must realize that (for large L) the sum defining $\Gamma(\lambda)$ is diverging [this is precisely the reason why it was not possible to recover $W_L(\rho)$ directly, and uniquely, from its moments given by Eqs. (5) and (6)]. However, the final equation (7) is perfectly sensible if $\Gamma_L(\lambda)$ is understood in its most general sense, namely, as the Laplace transform of the distribution $W_L(\rho)$, i.e., $\Gamma_L(\lambda)=\int_0^\infty d\rho\, e^{-\lambda\rho}W_L(\rho)$. In fact, one can check immediately that Eq. (7) is just the Laplace transform of Eq. (2). Thus, Eq. (7) for the generating function contains exactly the same information as Eq. (2) for the distribution function and, in particular, it enables one to obtain the limiting distribution $W^*(\rho)$ [Eq. (3)].

Let us now return to the "real" problem of obtaining the conductance distribution, at the mobility edge, from the cumulants given by Eq. (1). This problem is mathematically not well posed since (for $L>l$) the high-order cumulants grow too fast with n. In this respect the situation is the same as for the toy model considered above. Guided by the experience with the toy-model one can proceed along the following route.

(i) The first step is to consider Eq. (1) as a solution of some underlying differential equations for the cumulants $C_n(L)$. Assuming that these equations contain only first derivatives with respect to $\zeta\equiv\ln L$ (see above), one can write down the following set of equations:

$$\frac{\partial C_n}{\partial\zeta}=(\epsilon n^2-2n)C_n+2n\epsilon C_{n-1}. \qquad (8)$$

One can check that high-order cumulants, i.e., with $n>n_0$ (where n_0 is the first integer larger than $2/\epsilon$), have for large L the behavior required by Eq. (1). Cumulants with $n<n_0$ approach, in the limit $L\to\infty$, some fixed values C_n^* given by the recursion relation $C_n^*=2\epsilon C_{n-1}^*/(2-\epsilon n)$. For $n\ll 1/\epsilon$ this gives $C_n^*=\epsilon C_{n-1}^*$, as required by Eq. (1). Intermediate cumulants (the ones with n smaller but not much smaller than n_0) contain high powers of ϵ and are negligibly small. (The actual values of the intermediate cumulants should not be taken too seriously; the point is that the results of Ref. 6 are of an asymptotic nature and do not permit one to obtain the exact values of either the intermediate cumulants or the number n_0.) Note that for $n=0$ Eq. (8) is satisfied by an arbitrary constant c. It is convenient to use this freedom and to choose $c=\epsilon^{-2}$. With this choice the recursion relation for C_n^* can be used starting from

$n=1$ (the averaged conductance then is $C_1^* = c\epsilon = \epsilon^{-1}$ which is indeed the correct result). Since, however, the zero-order cumulant C_0 must be equal to zero (normalization), one should remember to subtract at the end the constant ϵ^{-2} from the cumulant generating function.

(ii) The second step is to multiply Eq. (8) by $(-\lambda)^n/n!$ and sum over n. This leads to the following equation for the cumulant generating function $G(\lambda) = \sum_n (-\lambda)^n C_n/n!$ [recall the remark following Eq. (7)]:

$$\frac{\partial G}{\partial \zeta} = \epsilon\lambda^2 \frac{\partial^2 G}{\partial \lambda^2} - (2-\epsilon)\lambda \frac{\partial G}{\partial \lambda} - 2\epsilon\lambda G\,. \qquad (9)$$

This equation contains all the information about the evolution of the cumulant generating function and, thus, of the conductance distribution, under change of the system size L.

Below I consider only the limiting solution of Eq. (9) which is obtained by equating the right-hand side to zero and solving the resulting ordinary differential equation. After subtracting the constant ϵ^{-2} one ends up with the following result for the cumulant generating function $G^*(\lambda)$ in the limit $L \to \infty$:

$$G^*(\lambda) = 2^{N+1} \frac{N^2}{(2N-1)!} \lambda^N K_{2N}(2\sqrt{2\lambda}) - N^2\,, \qquad (10)$$

where K is a modified Bessel function and it was assumed, for simplicity that $1/\epsilon \equiv N$ is integer. One can check, by direct differentiation of $G^*(\lambda)$ at $\lambda = 0$, that Eq. (10) reproduces correctly the low-order cumulants C_n^* (for $n \le 2N-1$), whereas the high-order cumulants $(n > 2N-1)$ diverge. The limiting distribution $P^*(g)$ is given by the inverse Laplace transform

$$P^*(g) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[G^*(\lambda) + g\lambda]\,d\lambda\,. \qquad (11)$$

Equation (11) is the main result of this Letter and it tells us that $P^*(g)$ is an absolutely universal function which depends only on $N \equiv 1/\epsilon$. The integral can be evaluated, approximately, for different regions of g. It turns out that for g near its average value $\bar{g} = N$ the distribution is nearly Gaussian, with a width $\langle \Delta g^2 \rangle \approx 1$:

$$P^*(g) \approx (1/\sqrt{2\pi}) \exp[-\tfrac{1}{2}(g-N)^2]\,, \qquad |g-N| \le (N \ln N)^{1/2}\,. \qquad (12)$$

For larger g, namely when $g - N \gg (N \ln N)^{1/2}$, $P^*(g)$ crosses over from Gaussian shape to a power law $P^*(g) \approx A_N g^{-2N-1}$ where the coefficient $A_N = N^2 2^{2N}/(2N-1)!$. It is this power-law decay that leads to divergent moments for $n > 2N-1$. Finally, it follows from Eqs. (10) and (11) that $P^*(g)$ has a term $\exp(-N^2)\delta(g)$; i.e., a small but finite fraction of the mobility edge ensemble (for $L \to \infty$) corresponds to strictly insulating samples.

Note in conclusion: (i) Using the cumulants calculated by Altshuler, Kravtsov, and Lerner,[6] it was possible to obtain the limiting distribution $P^*(g)$ for an ensemble of samples at the mobility edge, in the $L \to \infty$ limit. In spite of the fact that high-order cumulants are nonuniversal (e.g., depend on the mean free path l) and grow indefinitely with L, the limiting distribution $P^*(g)$ is absolutely universal and depends only on dimensionality. Let us emphasize, however, that the problem of finding the distribution from the cumulants given by Eq. (1) is mathematically not well posed (for $L > l$), so that certain plausible arguments were involved in going from cumulants to the distribution. (ii) Limiting distributions exist also away from the mobility edge: On the metallic side of the transition the conductance distribution $P_L(g)$, for large L, approaches a Gaussian, while on the insulating side there are strong reasons to believe that the distribution for the variable $x \equiv \ln g$ approaches a Gaussian shape.[5-7] (iii) In the context of renormalization and scaling one can say that any initial (bare) distribution $P_0(g)$ renormalizes (or scales) towards one of the three limiting distributions: metal, insulator, or the mobility edge distribution. The first two correspond to the trivial fixed points of the theory, whereas the mobility edge distribution $P^*(g)$ describes the nontrivial fixed point. Let us emphasize that, even though the distribution as a whole is renormalizable, the set of moments of the distribution is not renormalizable.[4,6] Thus, the moments (or cumulants) are not "good" scaling variables and do not reveal the universality of the problem (for a detailed discussion and relation to scaling see Refs. 5 and 7).[13] (iv) The calculations have been limited to the case when the local disorder was weak. Therefore the distribution $P^*(g)$ at the mobility edge could be studied only in $2+\epsilon$ dimensions. It is not clear to which extent the extrapolation to $\epsilon = 1$ is valid.

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[1]B. L. Altshuler, Pis'ma Zh. Eksp. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)]; P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).

[2]L. B. Ioffe, I. R. Sagdeev, and V. M. Vinokur, J. Phys. C **18**, L641 (1985).

[3]N. Kumar and A. M. Jayannavar, J. Phys. C **19**, L85 (1986).

[4]B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 342 (1986) [JETP Lett. **43**, 441 (1986)]; Zh. Eksp. Teor. Fiz. **91**, 2276 (1986) [Sov. Phys. JETP **64**, 1352 (1986)].

[5]B. Shapiro, Phys. Rev. B **34**, 4394 (1986); Philos. Mag. B **56**, 1031 (1987).

[6]B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, Phys. Lett. A **134**, 488 (1989); NORDITA report, 1989 (to be published).

[7]A. Cohen, Y. Roth, and B. Shapiro, Phys. Rev. B **38**, 12125 (1988).

[8]J. L. Pichard, N. Zanon, Y. Imry, and A. D. Stone, J. Phys. (Paris) **51**, 587 (1990); K. Muttalib, J.-L. Pichard, and A. D. Stone, Phys. Rev. Lett. **59**, 2475 (1987); Y. Imry, Europhys. Lett. **1**, 249 (1986); P. A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (N.Y.) **181**, 290 (1988); P. A. Mello, Phys. Rev. Lett. **60**, 1089 (1988).

[9]E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[10]H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).

[11]The calculations in Ref. 5 were somewhat more accurate (since the ϵ^2 term in ρ_c was kept) and led to $-\frac{4}{3}$, instead of -1, in the exponent of $(1+\rho)$ in Eq. (3). Our purpose here is not to get involved with the details of the model of Ref. 5 but rather to consider an instructive mathematical example of an equation [Eq. (2)] which describes the evolution of $W_L(\rho)$ towards the universal limiting function.

[12]See, e.g., N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, London, 1965).

[13]It is worthwhile to note that renormalization in statistical mechanics can also be viewed as renormalizing a probability distribution [see M. Cassandro and G. Jona-Lasinio, Adv. Phys. **27**, 913 (1978)].