Excitation Structure of the Hierarchy Scheme in the Fractional Quantum Hall Effect

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The hierarchy schemes for the fractional quantum Hall effect are reexamined and it is shown that different schemes all give the same lattice of excitations whose statistics is determined by the norm of the corresponding vector, and hence have equivalent Ginzburg-Landau theories. Similar ideas apply to the anyon liquid. The schemes can be generalized by using different lattices; many inequivalent states can be obtained at any filling factor (or value of the statistics parameter).

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Laughlin's wave functions[1] have won widespread acceptance as good model wave functions for the fractional quantum Hall effect[2] (FQHE) at filling factor $\nu=1/q$, but the situation at most other filling factors (with the exception of those related to $1/q$ by particle-hole conjugation or by filling of lower Landau levels) is somewhat less clear. Numerous schemes for extending Laughlin's ideas have been proposed, in particular what I will call the "standard" hierarchy,[3,4] a "variant" hierarchy,[5] and recently a "new" hierarchy.[6] These hierarchies are supported by different physical arguments but are alike in producing a ground state for every fraction $\nu=p/q<1$ such that q is odd, and in having fractionally charged excitations $\pm e^*=\pm e/q$.

Questions about the detailed structure of the excitation spectrum for filling factor ν have recently arisen because of its relevance to gapless excitations of an incompressible bulk FQHE state.[7] One may ask whether the hierarchies make equivalent predictions, whether there are other physically distinct incompressible states at the same ν, and how the Ginzburg-Landau (GL) theory, for $\nu=1/q$,[8-10] can be properly extended to other fillings. Similar questions may be raised about the ground states of a liquid of anyons.[11,12]

The main results of this paper are as follows. Incompressible FQHE systems will be regarded as equivalent when their filling factors are equal and they possess excitations whose quantum numbers and statistics correspond. (i) The quantum numbers of the possible "charged" excitations lie on a lattice of points in r-dimensional space for r levels of the standard hierarchy. Excitations of the same physical charge all have the same (fractional) statistics. All the hierarchy schemes are equivalent in this sense; different constructions involve different bases for the same lattice. This characterizes these systems nonhierarchically. (ii) The order parameter has r components, and the GL theory also involves r gauge potentials[12] and its structure is determined by the same lattice as the excitations. (iii) The constructions can be generalized further, in a basis-independent way, by using an arbitrary lattice, subject to certain rules. This produces other inequivalent states at *any* rational ν. (iv) Similar observations apply to spin singlet and partially polarized states, and to states of an anyon liquid.

I begin by writing the standard hierarchy electron wave function[3,4] in the form

$$\Psi(\{z_{0i}\})=\int\prod_{\alpha=1}^{r-1}\prod_{i=1}^{N_\alpha}d^2z_{\alpha i}\exp\left(-\frac{1}{4}\sum_i|z_{0i}|^2\right)\prod_{\alpha=0}^{r-1}\left[\prod_{i<j}(z_{\alpha i}-z_{\alpha j})^{a_\alpha}\prod_{ij}(z_{\alpha+1,i}-z_{\alpha j})^{b_{\alpha,\alpha+1}}\right]. \quad (1)$$

Equation (1) describes $N=N_0$ electrons at positions $z_i=z_{0i}$ and the integrals are over coordinates of quasiparticles at level $\alpha=1,\ldots,r-1$ in the hierarchy; the system contains N_α quasiparticles of level α at positions $z_{\alpha i}$. In the exponents, $a_0>0$ is odd, a_α $(\alpha>0)$ is even, $b_{\alpha,\alpha+1}=\pm1$, and $b_{r-1,r}=0$. Negative exponents are unconventional; quasiholes in the electron system couple with $b_{01}=1$ as usual, but quasielectrons couple with $b_{01}=-1$. This is an acceptable alternative to the usual Laughlin quasielectron or other proposals as long as the singularity at the center is removed by projecting onto holomorphic (lowest-Landau-level) functions. Such projection only introduces a short-range interaction into the effective many-component Coulomb plasma, described below. Alternatively, the factors with negative exponents may be replaced by positive powers of the complex-conjugate factor, times additional exponential factors. This freedom of choice in the hierarchy wave functions has no influence on the following; the above form makes the structure especially clear.

In order to work with states like (1), one needs to make an orthogonality postulate. To take overlaps of two many-quasiparticle states, one must integrate over the electron coordinates. One hopes that this makes the overlap vanish unless the positions of the $\alpha=1$ quasiparticles in one state nearly coincides with those in the other. If so, then the integrations in (1) for each state can be reduced to a single set of integrals for $\alpha=1$, and the process can be iterated. For a few well-separated $\alpha=1$ quasiparticles, this can be demonstrated explicitly,[13] and so should hold for $|a_\alpha|$ large. We will assume, as is standard, that it also holds for $|a_\alpha|$ as small as 2. Then expectations in (1) behave like those of a multicomponent generalization of Laughlin's Coulomb plasma.

A homogeneous ground state in the shape of a disk is obtained if

$$a_\alpha(N_\alpha-1)+b_{\alpha,\alpha+1}N_{\alpha+1}+b_{\alpha-1,\alpha}N_{\alpha-1}=0\,, \qquad (2)$$

for $\alpha=0,\ldots,r-1$, where $N_r=0$, $b_{-1,0}N_{-1}=-N_\phi$, and N_ϕ is the total physical flux in the area covered by the disk. Equations (2) state that charge neutrality is satisfied (including the background $-N_\phi$) in the multicomponent Coulomb gas (1). The filling factor is given by

$$\nu=\frac{N}{N_\phi}=\cfrac{1}{a_0-\cfrac{b_{01}^2}{a_1-\cfrac{b_{12}^2}{\ddots-\cfrac{b_{r-2,r-1}^2}{a_{r-1}}}}}\equiv\frac{p}{q}\,. \qquad (3)$$

Since a_0 is odd and positive, a_α, $\alpha>0$, are even and of either sign, and $b_{\alpha,\alpha+1}=\pm 1$, these give all the standard fractions; i.e., q is odd and p,q have no common factor.

Strengths of the logarithmic interactions in the Coulomb plasma resulting from (1) are given by the elements of

$$(G_{\alpha\beta})=\begin{pmatrix} a_0 & b_{01} & 0 & \cdots & \\ b_{01} & a_1 & b_{12} & & \\ 0 & b_{12} & a_2 & & \\ \vdots & & & \ddots & \\ & & & & a_{r-1}\end{pmatrix}. \qquad (4)$$

Then (2) becomes (neglecting 1 with respect to N_α)

$$(G_{\alpha\beta}N_\beta)=\begin{pmatrix} N_\phi \\ 0 \\ \vdots \\ 0\end{pmatrix} \qquad (5)$$

and by inversion of (5)

$$\nu=(G^{-1})_{00}=\det G'/\det G \qquad (6)$$

by Cramer's rule, where G' is the $(r-1)\times(r-1)$ matrix with elements $G_{\alpha\beta}'=G_{\alpha\beta}$ for $\alpha,\beta>0$.

A quasiparticle at z may be obtained by inserting $\prod_{\alpha i}(z_{\alpha i}-z)^{f_\alpha}$ with f_α integers into (1). The "fluxes" (or strictly, vorticities) f_α are screened by the generalized Coulomb plasma, producing screening "charges" δN_α *locally* around z,

$$G_{\alpha\beta}\delta N_\beta=-f_\alpha\,. \qquad (7)$$

Note that the physical electron number $\delta N=\delta N_0$ but f_0 is not the total effective physical flux because the quasiparticles δN_α constitute a backflow.

The statistics of the excitations can be found by generalizing the method of Arovas, Schrieffer, and Wilczek.[14] The phase $e^{i\theta}$ obtained by interchanging two identical excitations is

$$\theta/\pi=-f_\alpha\delta N_\alpha$$
$$=f_\alpha(G^{-1})_{\alpha\beta}f_\beta=\delta N_\alpha G_{\alpha\beta}\delta N_\beta\,. \qquad (8)$$

Here the direction of interchange is fixed for all ν by demanding that $\theta/\pi=1/q$ for a quasihole in the Laughlin state.[14] For a charge $\delta N=\pm 1/q$ excitation in the standard hierarchy

$$\frac{\theta}{\pi}=\cfrac{1}{a_{r-1}-\cfrac{b_{r-2,r-1}^2}{\ddots-\cfrac{b_{01}^2}{a_0}}}\,, \qquad (9)$$

which can also be obtained from Halperin's equations.[4] Some properties of this expression are given elsewhere.[15] That (9) is independent of the type of excitation will be confirmed below.

The same calculation also gives Berry's phase per unit area due to the effective background magnetic field seen by the excitation as $-f_\alpha\bar\rho_\alpha=\delta N/2\pi$, where from (5) $\bar\rho_\alpha=(G^{-1})_{\alpha 0}/2\pi$ are the average densities and so only excitations with nonzero physical charge see a field, which is the physical field, as one might have expected. These excitations therefore have Landau-level-type spectra, while the neutral excitations are propagating waves.

The set of possible excitations $\{(f_\alpha)|f_\alpha\in\mathbb{Z}\}$ may be regarded as lying on an "excitation lattice" Λ^* in a space $\mathbb{R}^r$. The coordinates f_α are the components of each lattice point in a basis $\mathbf{e}_\alpha^*$, $\alpha=0,\ldots,r-1$, of Λ^* whose *Gram matrix*[16] of scalar products is $(G^{-1})_{\alpha\beta}=\mathbf{e}_\alpha^*\cdot\mathbf{e}_\beta^*$. Thus θ/π is just the "squared length" (norm) of a vector in the lattice (not necessarily positive since G^{-1} is not necessarily positive definite). A transformation $\mathbf{e}_\alpha^*\to\mathbf{e}_\alpha^{*\prime}=S_{\alpha\beta}\mathbf{e}_\beta^*$ with S having integer matrix elements and determinant 1 changes the basis from $\mathbf{e}_\alpha^*$ to $\mathbf{e}_\alpha^{*\prime}$ but leaves the structure invariant.

For excitations (f_α) such that (δN_α) are all integers, one sees that the wave function is that obtained by adding or subtracting electrons or quasiparticles at z. Thus, as for Laughlin's states,[9] such combinations of fluxes are equivalent to adding or removing particles. Therefore a composite operator which adds such fluxes and compensating (quasi)particles has no net charge δN_α and exhibits long-range order; it is an *order parameter.* Pure states, with nonvanishing order-parameter expectations, are constructed[9] by taking linear combinations of states of different N_α with definite phases θ_α. Fluctuations in N_α change the quasiparticle distribution at the edge but leave the filling factor in the bulk unchanged.

The order parameters are in one-to-one correspondence with the integer-charged excitations that they contain, which form an r-dimensional sublattice Λ of Λ^*, which I call the "condensate lattice." By (8), the Gram matrix of Λ is G, so all scalar products of vectors in Λ are integers; i.e., Λ is an *integral lattice*. Λ^* is the dual lattice of Λ since it has the inverse Gram matrix,[16] and becomes an integral lattice if rescaled by $\sqrt{q}=\sqrt{\det G}$. The sublattice $\Lambda^\perp$ consisting of vectors of Λ having zero physical charge ($\delta N=\delta N_0=0$ in the original basis) has Gram matrix G'. $\Lambda^\perp$ is an *even* lattice (norms of these vectors are even because a_α are even for $\alpha>0$), and so these excitations have Bose statistics. It is easy to show from the form of (4) that they exhaust the neutral excitations, i.e., $(\Lambda^*)^\perp=\Lambda^\perp$, and hence that in the standard hierarchy the statistics of an excitation depends only on its charge δN.

These results imply that the form of the GL action[8-10] must be $S=\int d^2x\,dt\,L$, with

$$L=\tfrac{1}{2}(\partial_\mu\theta_\alpha-A_\mu\delta_{\alpha,0}-\mathcal{A}_{\mu\alpha})C^{\mu\nu}_{\alpha\beta}(\partial_\nu\theta_\beta-A_\nu\delta_{\beta,0}-\mathcal{A}_{\nu\beta})$$
$$+\bar{\rho}_\alpha(\partial_0\theta_\alpha-A_0\delta_{\alpha,0}-\mathcal{A}_{0\alpha})$$
$$+\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}\mathcal{A}_{\mu\alpha}(G^{-1})_{\alpha\beta}\partial_\nu\mathcal{A}_{\lambda\beta}\,, \qquad (10)$$

where $\mu,\nu,\lambda=0,1,2$ are space-time indices, the A_μ are the physical gauge potentials, $\mathcal{A}_{\mu\alpha}$ are internal gauge potentials, $C^{\mu\nu}_{\alpha\beta}=\eta^{\mu\nu}C^{(\mu)}_{\alpha\beta}$, $\eta^{\mu\nu}=\mathrm{diag}(1,-1,-1)$, and $C^{(\mu)}$ are arbitrary positive-definite matrices. The θ_α can be regarded as coordinates on a torus $\mathbb{R}^r/2\pi\Lambda^*$ and hence vortices are labeled by their flux $\int d^2x\,\nabla\times(\mathcal{A}_\alpha+A\delta_{\alpha,0})=2\pi f_\alpha$ and (7) and (8) follow from the final Chern-Simons term in (10). The second term gives the effective fields.

Jain's first construction[6] used wave functions χ_r for r filled Landau levels (LLs):

$$\chi_{(n_1+1/r_1)^{-1}}=(\chi_1)^{n_1}\chi_{r_1}\,, \qquad (11)$$

where $\nu=(n_1+1/r_1)^{-1}$ and n_1 is even. Even for $r_1>1$, (11) has nonzero projection to the lowest LL when $n_1>0$, the $\bar{z}$'s becoming $\partial/\partial z$'s. The resulting state may be described in terms of "fictitious LL's" or by saying that the electrons have been divided into r_1 species, each species having a different number of $\bar{z}$ factors for each electron. Thus the wave function is very close in form to a multicomponent Coulomb plasma (for χ_r itself we have r decoupled Coulomb plasmas and so the GL theory for $\nu=r$ is r copies of that for $\nu=1$). Excitations can be made by introducing holes into a single fictitious LL (or inverse powers to obtain quasielectrons). The different LL quasiholes are orthogonal in the thermodynamic limit, by a Coulomb-gas calculation, because the large number of factors of the form (z_i-z) act on distinct sets of particles. In fact, such arguments show[17] that the system exhibits a *spontaneous breakdown of permutation symmetry* and one can ignore the antisymmetrization of electrons among the species. Consequently, the $\bar{z}$ factors can be omitted and the system behaves just as an r_1 component Coulomb plasma, in which the Gram matrix G clearly has diagonal elements n_1+1 and off-diagonal n_1. These entries refer to a basis $\mathbf{e}_\alpha$ for Λ of equally charged excitations $\delta N=-1$ so the basis order parameters consist of one added electron and one of the flux combinations $\mathbf{e}_\alpha$.[18]

To make contact with the standard hierarchy, I now change basis. As the first basis vector take $\mathbf{e}_0$ which has norm n_1+1. For the remainder take $\mathbf{e}'_\alpha=\mathbf{e}_\alpha-\mathbf{e}_{\alpha-1}$, $\alpha=1,\ldots,r-1$, which have $\delta N=0$ and norm 2. The off-diagonal scalar products give -1 for adjacent members of the sequence and zero otherwise. The new Gram matrix is therefore tridiagonal like (4), proving that quantum numbers and statistics of excitations are the same as those of the standard hierarchy at the same filling factor (as can also be shown by direct calculation of Λ^*). $\Lambda^\perp$ is here the root lattice A_{r-1} of SU(r),[16] and the r species behave as the fundamental representation of this group, though there is no reason why the Hamiltonian should respect all of this symmetry.

Another set of filling factors $\nu=(n_1-r_1^{-1})^{-1}$, $r_1>1$, is obtained using the conjugate of χ_{r_1} in (11), or powers n_1-1, n_1 in the Coulomb plasma, and leads in the hierarchy basis to -2 in place of 2 in G; SU(r) "symmetry" is still present. Jain has emphasized[6] that these two families include most of the experimentally observed filling factors.

Given a state χ_ν, a new filling factor is obtained[6] by adding electrons in new fictitious LLs and then attaching flux to all the particles:

$$\chi_\nu\rightarrow\chi_{\nu'}=(\chi_1)^n\chi_{r+\nu}\,, \qquad (12)$$

where n is even and $\nu'=[n+1/(r+\nu)]^{-1}$, giving a "new" hierarchy of states labeled by sequences $n_1,r_1,n_2,r_2,\ldots,n_k,r_k$ for k steps. Once again there is a basis for Λ of $\delta N=-1$ excitations, one for each of the $r=\sum_{l=1}^k r_l$ species. Now take $\mathbf{e}_0$ to be one of the last set of r_k fluxes, and the $\mathbf{e}'_\alpha$ to be differences of the $\delta N=-1$ basis vectors, working back down the hierarchy. The resulting tridiagonal Gram matrix has diagonal n_k+1, 2 (r_k-1 times), $n_{k-1}+2$, 2 ($r_{k-1}-1$ times), ..., 2, and off-diagonal elements -1, which is the standard hierarchy form (4). Including negative entries in $n_1,\ldots,r_k$ gives all the standard hierarchy states.

The variant hierarchy[5] is sufficiently similar to the standard one not to require separate discussion here; it again produces the same lattices Λ^* of excitations.

The hierarchy construction can be generalized by taking an arbitrary Gram matrix G, whose matrix elements specify a ground state as in (1). In this basis, G_{00} must be odd because of Fermi statistics, and the other diagonal elements even, and so $\Lambda^\perp$ is even. Inequivalent lattices give inequivalent FQH states. Then $\nu=p/q$ where $q=\det G$ and $p=\det G'$ may have common factors. Note that ν need not have odd denominator. Equations (5), (7), (8), and (10) continue to hold. This very large set of possible states is just those having a basis of order parameters containing a single electron since a basis for Λ of $\delta N=-1$ (or a Jain-type construction) can always be obtained. An elegant example is obtained by replacing

G' by the Gram matrix of D_{r-1}, the root lattice of $SO(2(r-1))$,[16] $r>4$. Taking $G_{00}=m$, odd and depending on how G' is extended to G, one can obtain $\nu=1/(m-1)$or $\nu=1/(m-2)$, and so reproduce $\nu=1/q$ but with a lattice of dimension r.

States with some or all of the spins of the electrons reversed can be handled similarly; one of the δN_a is identified as δS^z. As examples, Halperin's $\nu=2/(2n+1)$ spin-singlet states[19] have the same lattice structure[15] as Jain's construction (11) for $r_1=2$, while a singlet state proposed by Jain[6] for $\nu=\frac{1}{2}$ is equivalent to that in Ref. 20.

The present results should shed light on the fractionally charged edge excitations.[7,21] Also, on surfaces of nontrivial topology, like the torus, general principles[15] imply a ground-state degeneracy[21] in the thermodynamic limit, the degeneracy being given by a factor $|\Lambda^*/\Lambda| = \det G = q$ for each "handle." For the hierarchy states, p,q have no common factors, so this is just the minimal degeneracy q for the torus found by Haldane.[22]

The hierarchy for the anyon liquid[12] parallels that for the FQHE for bosons (for which Λ is even) with the statistics parameter $\alpha_s=\theta/\pi$ playing the role of ν; I find that the space of order parameters is $r+1$ dimensional for an r-level fraction.

In conclusion, I have shown the existence of previously unnoticed structure in the hierarchy schemes which characterizes these states completely at the GL level. This classifies all states having only single-electron condensates.

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