Universal Critical-Point Amplitudes in Parallel-Plate Geometries

John L. Cardy
Department of Physics, University of California, Santa Barbara, California 93106
(Received 22 June 1990)

In a critical system contained between parallel plates, it is known that the magnetization and energy density profiles close to one wall are modified due to the presence of the other wall. In addition, there is a long-range force between the walls. It is shown that the ratios of the amplitudes governing these effects are universal, and moreover, independent of the boundary conditions at either wall. Their values are proportional to the scaling dimension of the density under consideration. Results are also given for the effects of local curvature of the wall on the density profiles. These results are shown to be a consequence of the conformal invariance of the critical system.

PACS numbers: 64.60.Fr, 05.70.Jk

Consider a d-dimensional generalization of a parallel-plate geometry in which two $(d-1)$-dimensional hyperplanes are separated by matter, for example, a binary fluid, at a bulk critical point. Some time ago, Fisher and de Gennes[1] argued that the reduced free energy per unit area of such a system should contain a term, depending on the separation D, of the form $A_{ab}D^{-d+1}$, where the amplitude A_{ab} was later argued to be universal,[2] depending only on the types of boundary conditions a and b at the respective walls. This term in the free energy gives rise to a Casimir- or van der Waals-like long-range force between the two plates. Subsequently, it was understood that the possible types of boundary conditions are related to universality classes of surface critical behavior.[3] In the most common example of a binary fluid, there is preferential absorption of one component, leading to a nonzero profile for the order parameter near the wall.

In the same paper, Fisher and de Gennes[1] gave arguments, based on a local form of the free energy in an inhomogeneous system, that, in such a situation, the order-parameter profile at a distance $z \ll D$ from one wall would be modified by the presence of the other wall by a relative amount $B_{ab}(z/D)^d$. The value of the exponent in this relation was confirmed in a renormalization-group analysis by Rudnick and Jasnow,[4] and for the two-dimensional Ising model by Au-Yang and Fisher.[5] In this Letter, it is shown that it follows simply from an understanding of the role of the stress-energy tensor in the short-distance expansion near the wall. Moreover, this analysis applies not only to the order parameter, but to any scaling density $\langle\phi(z)\rangle$ which has a nontrivial profile near the wall. In addition, the ratio of the amplitudes B_{ab}^{ϕ} and A_{ab} is given by

$$\frac{B_{ab}^{\phi}}{A_{ab}} = -\frac{2^{d-1}d\pi^{d/2}}{\Gamma(d/2)}\frac{x_\phi}{c}, \qquad (1)$$

where x_ϕ is the scaling dimension of ϕ, and the universal number c, to be defined blow, is a generalization to d dimensions of the central charge, or conformal anomaly number.[6] While this number does not so far appear to be easily accessible in other experimental situations,[7] Eq. (1) does show that the ratio on the left-hand side is independent of the particular boundary conditions on either wall, and that the ratio of the amplitudes B_{ab}^{ϕ} for different scaling densities (for example, the magnetization and energy density) is simply given by the ratio of their scaling dimensions.

The relation of Eq. (1) was first noted by Burkhardt[8] for a certain class of scaling operators in two dimensions, in the course of the explicit calculation of their exact profiles in the finite geometry. It also follows rather simply for any operator in $d=2$ in the case where $a=b$, where it is found that both[9] $A_{aa}=-\pi c/24$ and[10] $B_{aa}^{\phi}=\frac{1}{6}\pi^2 x_\phi$ are in fact independent of the type of boundary condition a. However, the argument for the validity of Eq. (1), which is summarized below, is completely general and should apply to all conformally invariant theories, in particular to those describing the large-wavelength fluctuations of critical systems below their upper critical dimensionality.

Although the details of the argument are somewhat cluttered by a proliferation of indices, in essence they are rather simple. First, recall some results of conformal field theory[6,11] and critical behavior at walls[3] in d dimensions. The stress-energy tensor $T_{\mu\nu}(r)$ is defined as the response of the reduced Hamiltonian (or action) $\mathcal{H}$ to a general infinitesimal coordinate transformation $r^\mu \to r^\mu + \alpha^\mu(r)$: $\delta\mathcal{H} = -S_d^{-1}\int\partial^\mu\alpha^\nu(r)T_{\mu\nu}(r)d^dr$. The factor $S_d=2\pi^{d/2}/\Gamma(d/2)$ is included for convenience. In a conformal field theory, $T_{\mu\nu}$ is conserved, symmetric, and traceless. The dimension x_ϕ of a scaling operator $\phi(r_1)$ then dictates[6] the most singular term in its operator-product expansion (OPE) with the stress-energy tensor (taking $r_1=0$ for simplicity):

$$T_{\mu\nu}(r)\phi(0) = a_\phi\frac{r_\mu r_\nu - (1/d)r^2\delta_{\mu\nu}}{r^{d+2}}\phi(0) + O(r^{-d+1}), \quad (2)$$

where $a_\phi = dx_\phi/(d-1)$.

Correlation functions of bulk operators like $\phi(r_1)$ in general become singular as their arguments approach the wall. In order to have a renormalized continuum theory with operators defined at the wall, it is necessary to perform additional renormalizations.[3,12] These *surface* operators $\psi_j^{(s)}$ have in general different scaling dimensions $x_j^{(s)}$ from those in the bulk, resulting in a set of surface exponents. The set of such operators is characterized by the particular surface universality class, which are labeled above by $a,b,\ldots$. Examples would be the ordinary, special, and extraordinary transitions (with either sign of the surface magnetization) in an Ising-like system. Bulk operators a distance z from the wall may be expressed in terms of the adjacent surface operators by an analog of the OPE:[13]

$$\phi(r_\parallel,z)=\sum_j b_j^\phi z^{-x_\phi+x_j^{(s)}}\psi_j(r_\parallel)\,, \tag{3}$$

where $r_\parallel$ is the component of r parallel to the wall, and j labels the different possible surface operators, which include the identity operator I and the stress-energy tensor. The stress-energy tensor is, however, a special case. Its correlation functions are regular as the argument of $T_{\mu\nu}$ approaches the surface, and therefore it does not need any new renormalization at the wall. Thus the "surface" stress-energy tensor is the same operator as in the bulk, with its canonical dimension d. However, at the wall it should satisfy the boundary condition[14] $T_{\parallel z}=0$. This follows from the requirement that no length scale is implicit in the boundary conditions, so that the system is invariant under reparametrizations of $r_\parallel$. It is equivalent to the assumption that the system is exactly at a fixed point of the renormalization group, with respect to both the bulk and the surface fluctuations. In a real system, at bulk criticality, departures from this picture over microscopic distances from the boundary are expected.[3]

For a general bulk operator, the most relevant terms appearing on the right-hand side of Eq. (3) are the identity operator I and the stress-energy tensor.[15] Rotational invariance in the $\parallel$ subspace implies that the only components which can appear are the $(d-1)$-dimensional trace T_{ii} and T_{zz}. Since the total trace $T_{\mu\mu}$ is zero, these are, however, proportional. Thus, normalizing ϕ so that $b_I=1$, the short-distance expansion, Eq. (3), may be written (taking $r_\parallel=0$)

$$\phi(z)=z^{-x_\phi}[I+b_T^\phi z^d T_{zz}(0)+\cdots]\,. \tag{4}$$

Now, take the expectation value of this equation in the parallel-plate geometry. Conservation of the stress tensor implies that $\langle T_{zz}(z)\rangle$ is independent of z and thus equal to its value on either wall. Since the stress tensor keeps its canonical dimension, this expectation value must be proportional to D^{-d}, giving the Fisher–de Gennes exponent,[1] as confirmed by Rudnick and Jasnow.[4] From the definition of the stress-energy tensor, $\langle T_{zz}\rangle$ represents the force per unit area between the plates. Thus if the reduced free energy F per unit area is A_{ab}/D^{d-1}, it follows that $\langle T_{zz}\rangle=-(d-1)S_dA_{ab}/D^d$. The Fisher–de Gennes amplitude B_{ab} is thus proportional to the coefficient A_{ab} of the Casimir effect. The ratio depends on the coefficient b_T^ϕ, which may be determined as follows.

Consider now the semi-infinite geometry, denoted by SI. Conformal invariance predicts the exact form of the correlation function $\langle T_{\mu\nu}(r)\phi(r_1)\rangle_{\rm SI}$ in this geometry. This may be seen by making an inversion about the image point $\bar r_1$, which sends the half-space into the interior of a hypersphere, and r_1 into its center. In this geometry, denoted by $\odot$, the form of the correlation function is dictated by rotational invariance and the OPE, Eq. (2), to be

$$\langle T_{\mu\nu}(r')\phi(0)\rangle_\odot=a_\phi\frac{r'_\mu r'_\nu-(1/d)r'^2\delta_{\mu\nu}}{r'^{d+2}}\langle\phi(0)\rangle_\odot\,, \tag{5}$$

where[10] $\langle\phi(0)\rangle_\odot=2^{x_\phi}|r_1-\bar r_1|^{x_\phi}$. Note that the normalization is fixed by the short-distance behavior as $r_1\to r$. Transforming this result back to the half-space, and taking the limit as $r_1,\bar r_1\to 0$,

$$\langle T_{\mu\nu}(r)\phi(r_1)\rangle_{\rm SI}\sim\frac{2^{x_\phi}a_\phi}{r^{2d}}Q_{\mu\lambda}(r)Q_{\nu\sigma}(r)\frac{(r_1-\bar r_1)_\lambda(r_1-\bar r_1)_\sigma-(1/d)|r_1-\bar r_1|^2\delta_{\lambda\sigma}}{|r_1-\bar r_1|^{x_\phi-d+2}}\,, \tag{6}$$

where $Q_{\mu\nu}(r)=\delta_{\mu\nu}-2r_\mu r_\nu/r^2$. However, the left-hand side may also be evaluated in this limit using the short-distance expansion, Eq. (4). Since $\langle T_{\mu\nu}\rangle=0$ in the half-space, this gives

$$\langle T_{\mu\nu}(r)\phi(r_1)\rangle_{\rm SI}\sim b_T^\phi 2^{x_\phi-d}|r_1-\bar r_1|^{d-x_\phi}\langle T_{\mu\nu}(r)T_{zz}(0)\rangle_{\rm SI}\,. \tag{7}$$

The two-point function of the stress-energy tensor in the half-space on the right-hand side may be found using similar methods to those described above for $\langle T\phi\rangle$. The result is rather simple: It is equal to the two-point function $\langle T_{\mu\nu}(r)T_{\lambda\sigma}(r_1)\rangle_{\mathbf{R}^d}$ in the full space $\mathbf{R}^d$, plus an image term $\langle T_{\mu\nu}(r)\bar T_{\lambda\sigma}(\bar r_1)\rangle_{\mathbf{R}^d}$, where $\bar T_{\lambda\sigma}$ is the reflection of $T_{\lambda\sigma}$, that is, the z components are reversed in sign. As $r_1\to 0$, these two terms become identical if $(\lambda\sigma)=(zz)$. The two-point function in the full space is determined by rotational symmetry, conservation, and tracelessness to have the form[6]

$$\langle T_{\mu\nu}(r)T_{\lambda\sigma}(0)\rangle_{\mathbf{R}^d}=\frac{c}{r^{2d}}\left[Q_{\mu\lambda}(r)Q_{\nu\sigma}(r)+Q_{\nu\lambda}(r)Q_{\mu\sigma}(r)-\frac{2}{d}\delta_{\mu\nu}\delta_{\lambda\sigma}\right]\,, \tag{8}$$

where the number c is universal. This is one generalization to $d \neq 2$ of the central charge which plays a ubiquitous role in two dimensions. Inserting this into Eq. (7), and comparing with Eq. (6), gives $b_T^o = 2^{d-2} a_o / c$, leading to the main result, Eq. (1).

One might ask whether the higher-order corrections to the behavior of $\langle \phi(z) \rangle$ also satisfy similarly simple relations. The answer is negative. Results from two dimensions[16] show that the higher-order terms in the short-distance expansion, Eq. (4), contain in general several operators with the same scaling dimension. Thus it is not possible to disentangle their respective coefficients by comparing with the appropriate term in the expansion of $\langle T\phi \rangle$.

The knowledge of the $\langle T\phi \rangle$ correlation function in the half-space may be used in another way to compute the effect of *curvature* of the wall on density profiles nearby. According to the definition of the stress-energy tensor, the change in such a density corresponding to an infinitesimal change $z \to z + f(r_\parallel)$ in the shape of the wall is

$$\delta\phi(r_1) = -S_d^{-1} \int \langle T_{zi}(r)\phi(r_1) \rangle_{\text{SI}}\, \partial_i f(r_\parallel) d^d r \,. \tag{9}$$

This integral may be transformed by the divergence theorem into one over the hyperplane $z=0$, which may then be evaluated by deforming the surface of integration into a small hypersphere surrounding r_1. The result contains two terms, one proportional to $f(r_1)$, which describes the change in $\langle \phi(r_1) \rangle$ due to its distance from the wall being modified by this amount, and a second, more interesting, term given by $\delta\langle\phi\rangle/\langle\phi\rangle = -[x_o/2(d-1)] \times z \nabla_\parallel^2 f$. To lowest order in f and its derivatives, this is proportional to the local mean curvature of the wall. Thus (specializing to $d=3$ for clarity) the density profile near a curved wall is

$$\langle \phi(z) \rangle \sim z^{-x_o} \left[1 - \frac{x_o z}{4} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + O((z/R_i)^2) \right], \tag{10}$$

where R_1 and R_2 are the principal radii of curvature. Although the derivation outlined above is valid only in the limit of weak curvature, it is expected that, when computed in renormalized perturbation theory, all such quantities will involve only geometric invariants formed from the induced metric on the wall and its intrinsic and extrinsic derivatives.[12,17] Equation (10) also agrees with exact results for the exterior and interior of a sphere, obtainable by a finite conformal mapping from the half-space.[10]

To summarize, a set of relations has been found between critical-point amplitudes in parallel-plate geometries and at curved boundaries which indicate that certain ratios depend only on bulk properties and are independent of the boundary conditions. This is probably the first practical application of the principle of conformal invariance to critical systems in three dimensions. It would be very interesting to see whether these predictions are testable in binary fluids or other systems, for example, dilute polymer chains in restricted geometries.

The author is grateful to T. Burkhardt for communicating his results in two dimensions prior to publication, and to D. Cannell and J. Rudnick for discussions. This work was supported by NSF Grant No. PHY 86-14185.

[1]M. E. Fisher and P.-G. de Gennes, C. R. Acad. Sci. Ser. B **287**, 207 (1978).

[2]V. Privman and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).

[3]See, for example, H. W. Diehl, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1986), Vol. 10.

[4]J. Rudnick and D. Jasnow, Phys. Rev. Lett. **49**, 1595 (1982).

[5]H. Au-Yang and M. E. Fisher, Phys. Rev. B **21**, 3956 (1980).

[6]J. L. Cardy, Nucl. Phys. **B290 [FS20]**, 355 (1987).

[7]In Ref. 6, this number is shown to determine the leading anisotropic corrections to *bulk* correlation functions in restricted geometries. By an extension of two-dimensional arguments (see, for example, J. L. Cardy and I. Peschel, Nucl. Phys. **B300 [FS22]**, 377 (1988)) it should also be related to the coefficient of a logarithmic term in the free energy of a critical finite-size system.

[8]T. W. Burkhardt and T. Xue (to be published).

[9]H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).

[10]T. W. Burkhardt and E. Eisenriegler, J. Phys. A **18**, L83 (1985).

[11]J. L. Cardy, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1987), Vol. 11.

[12]K. Symanzik, Nucl. Phys. **B190 [FS3]**, 1 (1981).

[13]H. W. Diehl and S. Dietrich, Z. Phys. B **42**, 65 (1981); **43**, 281(E) (1981).

[14]J. L. Cardy, Nucl. Phys. **B240 [FS12]**, 514 (1984).

[15]In principle, other operators with scaling dimension less than d could also occur. Calculations based on the ϵ expansion (Ref. 3) and in $d=2$ (J.L. Cardy, Nucl. Phys. **B275 [FS17]**, 200 (1986)) show that this does not happen for the case of the most interest, the extraordinary transition. Such terms probably do occur at the ordinary transition at bulk multicritical points.

[16]A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).

[17]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, Cambridge, 1982).