# Self-Organized Criticality: Goldstone Modes and Their Interactions

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It is shown that critical behavior of systems with self-organized criticality can be explained as a Goldstone-mode phenomenon. The nonlinear interaction of the Goldstone mode causes the nontrivial critical exponents. Two more models are introduced with self-organized criticality features.

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The recently described[1] phenomena dubbed *self-organized criticality* are intensively studied now using analytical, experimental, and numerical methods.[2-15] The interest in these phenomena is caused by the large variety of systems in nature which exhibit self-similar behavior in their temporal or spatial fluctuations. The analogy with the second-order phase transition often works as a guideline in understanding the self-organized criticality (SOC) phenomena. There is no doubt now that there is a variety of different universality classes with different sets of exponents, some of them looking like mean field.[11] Scaling relations can be written for these exponents. It is also known that the upper critical dimension is $d_c=4$ for an isotropical model[3,5] and $d_c=3$ for a directed model.[12] These predictions were supported numerically.[12,15] But working with the second-order phase-transition analogy, there always remain some open questions: Why is the system exactly at the critical point? Why are these systems so common? What is the special reason for that?

In this paper we shall try to give a new viewpoint to the SOC phenomena as a natural property of Goldstone modes in a many-body degenerate state. First, we shall consider the properties of correlation functions in SOC models. We shall point out the striking analogy between these properties and properties of transverse fluctuations of order parameter in the $n$-component Heisenberg model below the Curie temperature. Later, we shall use this analogy to introduce two more models with SOC features.

Let us consider a few examples of SOC phenomena: (1) The relaxation process at nonzero temperature in a lattice-gas model after adding a new particle. It can be described as a diffusion of excess mass to the periphery of the system. (1a) If the diffusion is linear, we have a trivial example of a model with a self-organized criticality. (1b) Nonlinear diffusion of particles due to their interactions. One of the versions of the later case was described and solved by Hwa and Kardar.[5] (2) The same system at $T=0$. There is no thermally activated relaxation in the system; the dynamics of the system is completely determined by microscopical dynamical rules. They can be different. (2a) One can introduce the rule that if the number of particles at any particular site exceeds some critical value $z_c$, then excess particles jump randomly to one of $n$ neighboring sites.[11] (2b) Another rule is possible: If the number of particles exceeds the critical value $z_c$, then $n$ particles are moved to the neighboring sites, one particle per neighbor.[1]

In all these models, we have an order parameter—the average density of particles per site. The total amount of particles is a conserved quantity during the evolution of these models.

In the first two models the magnitude of the order parameter is determined by the boundary conditions. In models (2a) and (2b) it is determined by dynamical rules: In model (2a) the number of particles per site would be exactly equal to $z_c$; in model (2b) it is equal to $z_c-(n-1)/2$. It is the essential feature of threshold dynamics that if the average particle concentration is less than the above values, the system does not exhibit the "*critical*" behavior. The sharpness of the threshold condition causes the sharpness of order-parameter attenuation and breaks the symmetry between particles and "*holes*." The evolution of all these models after the adding of a new particle can be understood in terms of diffusional (or deterministic) propagation of excess density wave. In all these models if one particle is added, one particle (in average) should leave the system. Thus the density-wave propagator should be gapless (in momentum presentation). It means that there is no exponential decay or exponential growth in this propagator. For models (1a) and (2a), the relaxation is trivial and can be described by the Gaussian correlation function $G=1/k^2$. In models (1b) and (2b) the nonlinearity should be taken into account. It is clear that the nonlinearity cannot reduce the number of particles. The linear dynamics of any of the above models results only in displacements of particles. The nonlinearity of the models can be described as an additional displacement, dependent on the local concentration. In terms of the perturbational diagram technique, the nonlinearity can be taken into account by many-particle diagrams with the interaction vertex containing the displacement operator $e^{\mathbf{a}\cdot\nabla}-1$.[3] In the anisotropic case, the leading term in this operator should be $\mathbf{a}\cdot\nabla$, where $\mathbf{a}$ is a unit vector in a preferred direction. In the isotropic case, when there is no preferred direction, the leading term should be proportional to $\nabla^2$. It can happen that the nonlinearity can result in slowing down of the process or in speeding it up. In this case the interaction can be expressed as some time-shift operator $e^{\tau\partial_t}-1$. The leading term is then $\tau\partial_t$.

It is well known that gradient-dependent interaction (or $k$ dependent in momentum presentation) is the intrinsic feature of interaction of acoustic waves in solid medium or spin waves in ferromagnets. This is the common property of the so-called Goldstone modes: In the long-wavelength limit these modes can be reduced to a homogeneous displacement of the sample or to a uniform rotation of the whole spin system, and the nonlinear interaction between them should vanish.

In many physical systems, serious attention has not been paid to the Goldstone-mode interaction because it leads to a nontrivial scaling behavior of the spectrum of the system only if the space dimension is less than or equal to 2. This is because symmetry restrictions permit only interactions of the form $\nabla^2\psi^4$. A well-known example[16] is the interaction of transverse spin waves of the two-dimensional $XY$ ferromagnet ($n=2$) below the critical point.[17]

It is also known that in the limit $n=0$ the Heisenberg ferromagnet model describes the statistics of polymer chain solution.[18] The number of transverse modes in this case is equal to $-1$. The nontrivial renormalizations due to transverse-mode interactions result in anomalous correlations functions in a two-dimensional polymer melt.[19]

At $n=1$, we have an Ising model and there are no transverse modes at all. But we can formally consider the transverse fluctuations of the general $n$-component model[20] in the $n\to 1$ limit. Thus at $n=1$, the transverse correlation function can be calculated. It does not contribute to the partition function of the $n=1$ model, but it can be useful in certain applications. For example, in the polymer-magnet analogy, the case $n=1$, $T<T_c$ describes the solution of ring polymers with the average length of order $(T_c-T)^{-1}$. In this case, the transverse correlation function describes the end-to-end correlations of a single linear polymer chain placed in that solution.

Returning to nonlinear models (1b) and (2b), we see that the broken symmetry between particles and holes permits the $\psi^3$ interaction vertices (with gradient terms) and this results in a *higher critical dimensionality* of these models. Probably this is why they are often thought of as systems which due to some unknown reason are exactly at the point of the second-order phase transition. Actually these models are definitely below the phase-transition point, and what we think is their nontrivial criticality is simply the effect of interaction of the gapless modes.

Summarizing the above consideration, we suggest that for any ordered or correlated many-body state we can introduce the gapless modes associated with the translational invariance or with the degeneracy of order parameter. Even if the system has no such degeneracy, it can be considered as a limit case of a more general system and appropriate Goldstone modes can be found. In this case they do not contribute to any measurable quantity of the model, but they can have a simple geometrical interpretation. Below we shall describe two more systems which exhibit the SOC behavior. We generated these systems just using the above ideas.

*Percolation.*— The bond percolation can be formally described by the $S\to 1$ limit of the $S$-component Potts model. This is a model with a discrete symmetry of order parameter and there are no Goldstone modes in it. But we can obtain them in a slightly modified model called the dynamical percolation model.[21] Let us consider the porous media where the thickness of walls between two neighboring pores varies at random. A fluid is pumped into one randomly chosen pore under the pressure $p$. The fluid can penetrate also into the neighboring pores if the pressure $p$ is strong enough to break the walls between them. Then fluid spreads over the system forming a finite or infinite cluster (if $p>p_c$, where $p_c$ is some threshold value). The statistics and correlation functions of pores filled with a fluid are the same as in bond percolation theory. But the cluster structure is different. It is a treelike structure of pores filled up with fluid and broken walls between them. It is important that they are not in any closed-loop configuration of broken bonds: If the spreading fluid finds some way to a certain pore, there is no need to find another. Making a small shift and starting to pump fluid from any other point which belongs to the same cluster, we shall get the same picture of pores filled with a fluid but a different treelike structure of broken walls. If we consider only the bonds (broken walls) which are present in the second tree but absent in the first, we shall see that they also form a tree. We shall call it the "*difference tree*" or "$d$ tree." At $p<p_c$ we shall have only finite clusters with a finite correlation length; the same correlation length will describe all $d$ trees connected with the same cluster. At $p>p_c$ we have an infinite cluster which in terms of percolation theory can be characterized only by the average density (order parameter) and by the density-density correlation function (longitudinal correlation function) with a finite correlation length. Nevertheless, the distribution of $d$ trees generated by shifting the position of the starting point must be powerlike and will be controlled only by the size of the system $L$. It can be understood in a simple way by considering the $d$ tree generated by the shift from point 1 to point 2 whose magnitude is of order $L$. The size of this tree $F$ should be of order $L^d$. Point 2 can be reached in $L^2$ unit steps along the branch of the first tree, which connects points 1 and 2. Thus this big tree can be decomposed as a sum of $L^2$ trees generated by unit steps. Among these trees would be the trees of all sizes $F$ starting from the unit size. But the average size of a $d$ tree generated by a unit shift should be at least $L^{d-2}$. Thus they can be described by the distribution function

$$\rho(F,L) \sim (1/F^{\tau})g(F/L^{\sigma}) \tag{1}$$

with two critical exponents $\tau, \sigma$ related through

$$(2-\tau)\sigma = d-2\,. \tag{2}$$

What is the upper critical dimension for the statistics of $d$ trees? Using the above suggestions about the gradient terms in the interaction vertex we can believe that the critical dimension is reduced by 2 with respect to the critical dimension $d_c=6$ of the percolation and dynamical percolation models. This conjecture can be verified by a simple proof. The exponent $\tau$ in a distribution function of random trees is $\tau=\frac{3}{2}$. The fractal dimension of random trees is 4; thus $\sigma=4$. Substituting these mean-field values into (2) we obtain $d_c=4$.

*Directed percolation.*—The directed percolation model describes the propagation or spreading phenomena in anisotropical media. For example, it can be used for description of the spreading of infection or fire in a tree garden affected by wind. If one tree is infected, it can infect the neighboring trees with probabilities $p(\mathbf{q})$, where $\mathbf{q}$ is the vector indicating the direction of the spreading of infection. For the sake of simplicity we can assume that the probability to infect the tree against wind direction is zero, and all other probabilities are equal to $p$. If $p$ exceeds some critical value $p_c$, the infection from one tree will spread in a certain cone of directions forming the infinite cluster of infected trees. The angle of the cone depends on the difference $p-p_c$. It is remarkable that near the cone the conditions for starting a new epidemic process by infecting any healthy tree are exactly critical [see Figs. 1(a) and 1(b)]. Creating one after another new epidemic processes by infecting new points above the cone, we obtain the new critical cone structure [see Fig. 1(c)]. The process of formation of this structure is somehow very similar to the process of formation of a critical sandpile by adding sand on its top. The size distribution of the epidemics will be the same as above but with a

$$(2-\tau)\sigma = d-1 \tag{3}$$

relation between the critical exponents. There appears additional anisotropy (preferred direction) for new avalanches. It is caused by the local slope of the cone formed by previous avalanches. This anisotropy decreases additionally the upper critical dimension by 1 (in the same way $d=5$ for the directed percolation model is lower than $d_c=6$ for isotropical percolation). Taking into account the possible gradient terms, we suggest that the upper critical dimension is $d_c=2$ for this model. This can also be checked easily. The exponent 5 for random directed trees is the same, $\tau=\frac{3}{2}$, but the mass of the directed tree scales with its largest size as $F \sim L^2$; thus $\sigma=2$. Using Eq. (3) we get $d_c=2$. It is interesting to note here that $d_c=2$ is simultaneously the upper and the lower critical dimension of the model. As in the previous case the criticality in this model appears only at $p>p_c$.

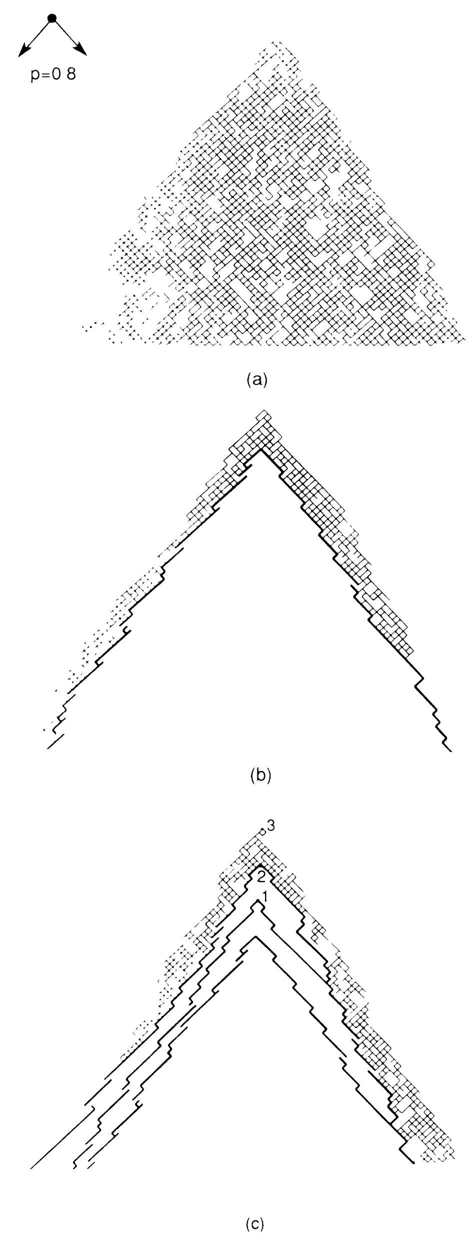


FIG. 1. (a) Spreading of a fire or infection in tree garden affected by wind. If one tree is infected, it can infect two neighboring trees with probability $p$. Here $p=0.8$; the infinite cluster of infected trees is shown. (b) Near the cone formed by "*dead*" trees the conditions for starting a new epidemic process by infecting any healthy tree are exactly critical. (c) Creating one after another new epidemic processes by infecting the points 1, 2, and 3, we obtain a new critical cone structure, similar to the previous ones.

The above models can help in understanding the possible self-organized criticality behavior of the "*life*" game, where the unit perturbations, in a steady state, create the restructuring pattern with a power-law distribution of their sizes.[8] The *life* game is a model with a stationary state and fixed average density. The structure of this state is similar to the structure of infinite cluster in the directed percolation model. What is quite different from the percolation cluster is that there must be other internal parameters characterizing the stationary state. These can be phases of stable blinking configurations, the parameters indicating their complexity, etc. So we can expect that the structure of a stationary state is described not only by the density parameter (as in percolation theory), but with some complex highly degenerate order parameter. In this case there can be gapless Goldstone modes (phasons). These phasons generated by a unit perturbation can be seen as a restructuring pattern with a power-law distribution.

One of the recent papers devoted to the SOC phenomena is entitled[22] *"Is the World on the Border of Chaos?"* Summarizing the above consideration we can answer this question: *"The World can only slide along this border."*

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[1]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988); J. Stat. Phys. **51**, 797 (1988); P. Bak and C. Tang, Phys. Today **42**, S27 (1989).

[2]Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).

[3]S. P. Obukhov, in *Random Fluctuations and Pattern Growth: Experiments and Models,* edited by H. E. Stanley and N. Ostrowsky (Kluwer Academic, Boston, 1989).

[4]M. H. Jensen, P. Bak, and K. Chen (to be published).

[5]T. Hwa and M. Kardar, Phys. Rev. Lett. **62**, 1813 (1989); P. Bak and C. Tang, J. Geophys. Res. **94**, 15635 (1989).

[6]A. Sornette and D. Sornette, Europhys. Lett. **9**, 192 (1989).

[7]J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).

[8]P. Bak, K. Chen, and M. Creutz, Nature (London) **342**, 780 (1989).

[9]P. Bak, K. Chen, and C. Tang (to be published).

[10]X. Che and H. Suhl, Phys. Rev. Lett. **64**, 1670 (1990).

[11]L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989).

[12]D. Dhar, Phys. Rev. Lett. **63**, 1659 (1989).

[13]D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).

[14]C. Tang, H. Nakanishi, and J. S. Langer, Phys. Rev. A **40**, 995 (1989).

[15]B. Grossmann, H. Guo, and M. Gzant, Phys. Rev. A **41**, 4195 (1990).

[16]*Note added.*—In a recent paper by D. Dhar (to be published), the similarity between the $n$-component Heisenberg model and SOC phenomena is also pointed out.

[17]A. Z. Patashinski and V. L. Pokrovski, *Fluctuation Theory of Phase Transitions* (Oxford Univ. Press, Oxford, 1975).

[18]J. Des Cloizeaux, J. Phys. (Paris) **36**, 281 (1975).

[19]E. S. Nikomarov and S. P. Obukhov, Zh. Eksp. Teor. Fiz. **80**, 650 (1981) [Sov. Phys. JETP **53**, 328 (1981)].

[20]A. M. Polyakov, Phys. Lett. **59B**, 29 (1975).

[21]C. DeDominicis, in *Dynamical Critical Phenomena and Related Topics,* edited by C. P. Enz, Lectures Notes in Physics Vol. 104 (Springer-Verlag, Berlin, 1979).

[22]P. Bak, in Proceedings of the Soviet-American Meeting on Frontiers in Condensed Matter Physics, New York, New York, 1989 (to be published).