Spectrum of a Passive Scalar in the Inertial-Convective Subrange of an Anisotropic Turbulent Flow

Harry F. Robey
University of California, Lawrence Livermore National Laboratory, P.O. Box 5508, L-487, Livermore, California 94550
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The spectral density of temperature fluctuations in an anisotropic, heated turbulent channel flow is studied experimentally using a new optical technique. The spectrum in the direction transverse to the flow shows an equilibrium subrange behavior characteristic of isotropic flows with a wave-number dependence of $\Phi_\theta(\kappa_y) \sim \kappa_y^{-11/3}$, whereas in the flow direction, $\Phi_\theta(\kappa_x) \sim \kappa_x^{-9/3}$. This difference is shown to result from the dominance of the mean strain rate over the turbulent strain rate field at low wave numbers, and the observed slopes are explained on dimensional grounds.

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In 1941, the concept of an "inertial subrange" was first introduced by Kolmogorov[1] to describe that portion of the energy spectrum $E(\kappa)$ of the turbulent velocity fluctuations in which energy was neither being input to the flow by external sources nor being dissipated by the effects of viscosity. By postulating the existence of an energy cascade in this subrange in which larger scales of motion transfer energy to smaller scales at a rate determined only by the rate of viscous dissipation ϵ, Kolmogorov showed that for a locally isotropic turbulent velocity field, the energy (per unit mass per unit wave number) should depend only on ϵ and the local wave-number magnitude as $E(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}$. This behavior is valid for that portion of the spectrum in which the wave number is less than the Kolmogorov microscale, $\kappa_d = (\epsilon/\nu^3)^{1/4}$, beyond which the effects of viscous dissipation become important. This $\kappa^{-5/3}$ dependence has been verified by numerous experiments where the assumption of local isotropy can be considered to be valid. The classic example is the experiment of Grant, Stewart, and Moilliet[2] in which the energy spectrum in a very-high-Reynolds-number tidal flow was measured over several decades in wave-number space.

Analogous results have been obtained concerning the spectra of dynamically passive scalars such as temperature or species concentration convected by a turbulent flow. Obukhov[3] and Corrsin[4] independently predicted the existence of an "inertial-convective" subrange in the spectrum of a passive scalar quantity θ, in direct analogy to Kolmogorov's inertial subrange for the velocity fluctuations. In the inertial-convective subrange, the power spectrum $\Gamma(\kappa)$ of the scalar fluctuations was shown to be of the form $\Gamma(\kappa) \sim \chi\epsilon^{-1/3}\kappa^{-5/3}$, where χ is the rate of dissipation of energy of the scalar fluctuations. This spectral behavior has been verified in the experiments of Gibson and Schwarz[5] for both temperature and concentration fluctuations in grid-generated turbulence and the experiments of Becker, Hottel, and Williams[6] which measured concentration fluctuations in a turbulent jet.

In the present paper, new results are presented which suggest that the inertial-convective subrange can be extended in a generalized form to lower wave numbers where the effect of the mean strain rate field of the flow generates structures of considerable anisotropy. This inherent anisotropy forces us to abandon the isotropic spectral functions $E(\kappa)$ and $\Gamma(\kappa)$ in favor of the three-dimensional spectral densities $\Phi_u(\boldsymbol{\kappa})$ and $\Phi_\theta(\boldsymbol{\kappa})$, which reflect the reality that the spectra for the lower-wave-number components must be a function of the coordinate direction.

The idea of extending the inertial-convective subrange to include the largest, energy containing eddies seems to have been first suggested by Heisenberg,[7] though without any discussion of anisotropy. Batchelor[8] also adopted this idea and employed the three-dimensional spectral density in studying the asymptotic behavior of the spectrum at the lowest wave numbers ($\kappa \to 0$), those containing very little energy. The region of the spectrum examined in the present paper, by contrast, concerns those eddies which do contain most of the energy and are large enough to be influenced directly by the mean strain rate of the flow.

The form of the spectral density $\Phi_\theta(\boldsymbol{\kappa})$ in the inertial-convective subrange can be rather simply formulated on dimensional grounds. The spectral density has dimensions of energy per unit mass per unit volume in wave-number space (L^5T^{-2}). Within the inertial-convective subrange, an equilibrium exists in which the rate of energy input to eddies of wave number $\boldsymbol{\kappa}$ is equal to the rate at which energy is transferred to smaller scales. This transfer of energy can occur either from the mean flow directly to the turbulence or by transfer from larger scales to smaller scales. In either case, the mechanism of energy exchange is the same. The eddies are exposed to a strain rate field which consists of an anisotropic mean component $S(\mathbf{x})$ and a locally isotropic turbulent component $s(\kappa)$. Energy is transferred when an eddy is aligned with the local strain rate vector and stretching of the vorticity field occurs. The strain rate has dimensions of T^{-1}, and its reciprocal is therefore the response time with which eddies equilibrate with the imposed strain rate field. The larger the strain rate, the shorter the time for equilibration.

In general terms, then, the spectral density can be formulated as

$$\Phi_\theta(\boldsymbol{\kappa}) \sim \frac{(\text{energy flux})(\text{response time})}{(\text{wave number})^3} \sim \frac{\chi s^{-1}}{\kappa^3}. \tag{1}$$

For $\Phi_\theta(\boldsymbol{\kappa})$, the energy flux is identified with the dissipation rate of the scalar fluctuation energy χ, whereas for $\Phi_u(\boldsymbol{\kappa})$, it is the viscous dissipation rate ϵ. The appropriate response time is determined by the nature of the velocity field. For a locally isotropic turbulent field, $s(\kappa) \sim \epsilon^{1/3}\kappa^{2/3}$ (see, for example, Tennekes and Lumley[9]). For κ large enough, $s(\kappa) \gg S(\mathbf{x})$ in all coordinate directions. The spectral density then depends only on wave-number magnitude as

$$\Phi_\theta(\kappa) \sim \frac{\chi(\epsilon^{1/3}\kappa^{2/3})^{-1}}{\kappa^3} \sim \chi\epsilon^{-1/3}\kappa^{-11/3}. \tag{2}$$

This can, of course, be directly compared with the original result of Obukhov and Corrsin by integrating over two wave-number dimensions, reproducing the one-dimensional spectral function Γ, which is only a function of wave-number magnitude:

$$\Gamma(\kappa) = \oint \oint \Phi_\theta(\kappa) d^2\kappa \sim \chi\epsilon^{-1/3}\kappa^{-5/3}. \tag{3}$$

At higher wave numbers it is known that $s(\kappa)$ cannot increase indefinitely with κ but is limited by viscosity to a maximum value of order $(\epsilon/\nu)^{1/2}$. Batchelor[10] showed that for this "viscous-convective" subrange in the case of small scalar diffusivity, $\Gamma(\kappa) \sim \chi\nu^{1/2}\epsilon^{-1/2}\kappa^{-1}$. The corresponding spectral density based on this limiting strain rate takes the form

$$\Phi_\theta(\kappa) \sim \frac{\chi(\nu^{-1/2}\epsilon^{1/2})^{-1}}{\kappa^3} \sim \chi\nu^{1/2}\epsilon^{-1/2}\kappa^{-9/3}. \tag{4}$$

Again, $\Gamma(\kappa)$ is related to $\Phi_\theta(\kappa)$ by a factor of κ^2, reproducing Batchelor's original result.

On the low-wave-number end of the inertial-convective subrange, we can apply the same dimensional reasoning. In this regime, the three-dimensional nature of the spectral density must be employed, since the appropriate strain rate now depends on the coordinate orientation. In a flow with pure strain, the orientation dependence is straightforward. If a maximum extensional strain stress exists in one direction (a contracting nozzle, for example), it is in that direction that the energy transfer from the mean flow occurs and in that direction that the vorticity field of the large eddies will be aligned. In a shear flow such as the present channel flow, however, there is a mean strain rate plus a mean rotation rate. The principal axes of the strain rate tensor are oriented at 45° to the flow direction and to the wall normal. The pure strain in this direction is the source of the energy transfer from the mean flow to the large eddies. The rotation rate of the flow then rotates vector components of vorticity oriented normal to the wall into the flow direction. The resulting large eddies are therefore oriented in the flow direction. Note that there is no corresponding first-order term which rotates longitudinal vorticity components away from the flow direction. This effectively allows a decomposition of $\Phi_\theta(\boldsymbol{\kappa})$ at low wave number into orthogonal components parallel and transverse to the flow direction, as mixing between these components occurs on a slower time scale based on $s^{-1}(\kappa)$.

In the flow direction x, as κ_x becomes small, $S \gg s(\kappa)$ and Eq. (1) becomes

$$\Phi_\theta(\kappa_x) \sim \chi S^{-1}/\kappa_x^3 \sim \chi S^{-1}\kappa_x^{-9/3}. \tag{5}$$

This defines a new equilibrium region of the spectrum, the anisotropic equilibrium range, where the largest eddies are in equilibrium with the mean strain rate field rather than with the turbulent strain rate field as is the case for the higher-wave-number part of the inertial-convective subrange.

In the directions transverse to the flow, however, there is no mean strain rate component. The strain rate field in this direction is simply the random component $s(\kappa)$. The spectral density in this direction is then the same as in Eq. (2). As far as eddies oriented transverse to the flow are concerned, the strain rate field they are exposed to is indistinguishable from that of an "isotropic" flow. The spectrum, therefore, has the same behavior in the inertial-convective subrange, *even in an anisotropic flow.*

The relations given in Eqs. (2) and (5) have recently been verified experimentally using a new optical technique which allows us for the first time to obtain $\Phi_\theta(\boldsymbol{\kappa})$ directly. The technique is described in detail in Refs. 11 and 12. In brief, a fully developed turbulent channel flow with a Reynolds number based on channel width, $\mathrm{Re}_D = 16\,000$, is traversed by a near-diffraction-limited laser beam. The initially uniphase beam acquires phase and amplitude variations which correspond to the refractive-index field, which in the present experiments is proportional to the fluctuating temperature field of the flow. The beam is brought to a focus with a positive lens and, as is well known from Fourier optics, produces an intensity distribution in the focal plane which is directly related to the Fourier transform of the 2D intensity distribution in the aperture of the lens. The coherent background illumination, which is many orders of magnitude larger than the scattered intensity field, is removed by exploiting the photorefractive properties of $BaTiO_3$ as a high-pass temporal filter.[13-15] The $BaTiO_3$ effectively scatters out the coherent portion of the beam and allows the incoherent portion corresponding to the spectral density of the turbulence to pass through unaltered.

The intensity distribution generated in the focal plane is directly proportional to the two components of $\Phi_\theta(\boldsymbol{\kappa})$

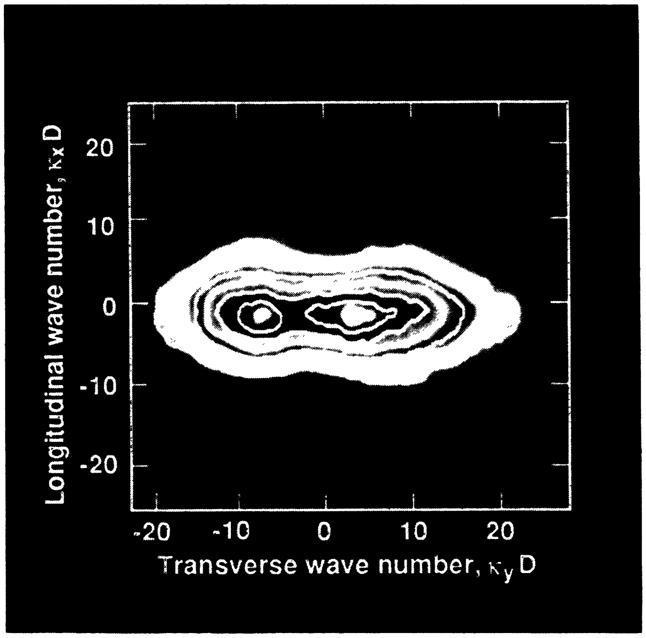

FIG. 1. Photograph and contours of 2D spectral density, $\Phi_\theta(\kappa_x,\kappa_y)$.

which are transverse to the beam propagation direction. In the present experiments, the flow is taken to be in the x direction and the beam propagates in the z direction. The components of $\Phi_\theta(\boldsymbol{\kappa})$ thus measured are those in the x and y directions (parallel and transverse to the flow). Figure 1 shows a digitized photograph of $\Phi_\theta(\kappa_x,\kappa_y)$ obtained with this technique. The two peaks in the spectrum at normalized wave numbers $\kappa_y D \approx \pm 2\pi$ correspond to the largest allowed transverse scales of motion which are of the order of the channel thickness D. Note that no such peak can be resolved in the flow direction, as the largest scales are significantly larger than the channel width D. Directional spectra in the x and y directions corresponding to Eqs. (2) and (5) are shown in Fig. 2. The transverse spectrum has a slope of approximately $-\frac{11}{3}$ and the longitudinal spectrum $-\frac{9}{3}$ as expected. The region of $\Phi_\theta(\boldsymbol{\kappa})$ shown in Fig. 2 is the low-wave-number portion. It is expected that for higher-wave-number components, the slope in the longitudinal spectrum will become asymptotic to $-\frac{11}{3}$ at a value of κ where $s(\kappa) \approx S$. Using the isotropic relations $s(\kappa) \sim \epsilon^{1/3}\kappa^{2/3}$, this transition will occur at $\kappa_{\rm iso} \sim S^{3/2}/\epsilon^{1/2}$. An increase in the value of the mean strain will therefore generate anisotropy down to smaller scales. This suggests that the turbulent spectrum will be of the form shown in Fig. 3, where the longitudinal spectra are shown normalized at high wave numbers. Increasing mean strain moves the peak in the spectrum toward lower wave numbers due to the increased vortex stretching of the largest eddies, and the isotropic cutoff moves toward higher wave numbers. Spectra of transverse components as well as the spectrum of isotropic turbulence are also contained in Fig. 3 as the limiting case of $S \rightarrow 0$. Though the present discussion pertains directly to the spectra of passive scalars, analogous results are expected for the low-wave-number behavior of the inertial subrange of turbulent velocity fluctuations as well.

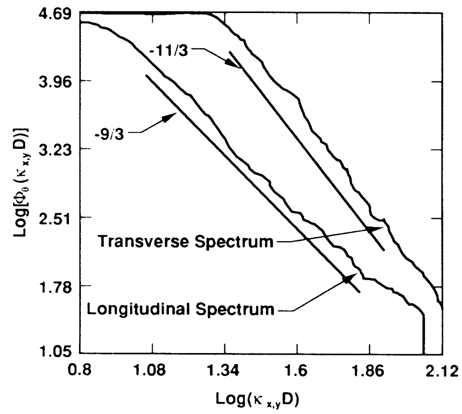

FIG. 2. Directional spectra in the transverse (y) and longitudinal (x) directions.

An interesting comparison with the present results can be made with the recent numerical results of Lesieur and Rogollo.[16,17] In their large-eddy simulations, the spectra of decaying isotropic turbulent velocity and temperature fields are studied. A κ^{-1} region is observed to develop in the low-wave-number portion of their 1D temperature spectrum (corresponding to the κ^{-3} region observed here for the longitudinal component of the spectral density). In their analysis, the form of the low-wave-number spectrum is described by an expression which is essentially identical to Eq. (1), where their response time is identified with the large-eddy turnover time, $\langle u^2\rangle/\epsilon$. They arrive at the same conclusion that the κ^{-1} region originates due to the straining by the large-scale motions. The difference between these numerical simulations and the present results, however, is that here the large-scale straining motion is identified with the mean strain rate of the flow, not the strain rate of the large-

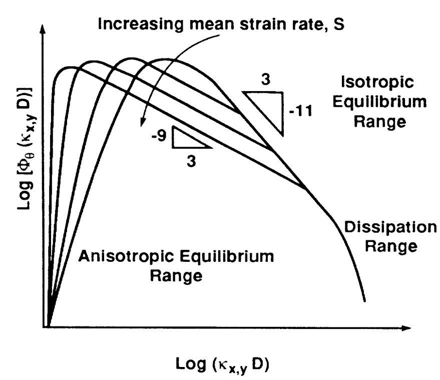

FIG. 3. Form of longitudinal spectra for increasing mean strain rate.

scale turbulent eddies. The fact that the transverse spectrum does not exhibit a κ^{-3} region in the spectral density indicates that the large-scale turbulent motions are not the dominant source of this low-wave-number behavior. In the absence of a mean shear, however, the strain field of the large eddies will be the dominant mechanism.

For the same reason, namely, the absence of a mean shear component, the large-eddy simulations do not show a corresponding κ^{-1} region in the spectrum of the velocity fluctuations, whereas the present analysis suggests that this should be the case. Previous studies such as the turbulent pipe flow measurements of Perry and Abell[18] and the atmospheric boundary layer measurements of Pond *et al.*[19] demonstrated that a κ^{-1} region does exist at low wave number for the longitudinal velocity spectrum. In neither of these studies, however, is there much discussion as to the origin of this behavior. The present technique with its unique ability to directionally resolve the spectra of anisotropic shear flows lends strong support to the idea that it is the transfer of energy directly from the mean strain rate of the flow that is responsible for this observed low-wave-number behavior in all of these studies.

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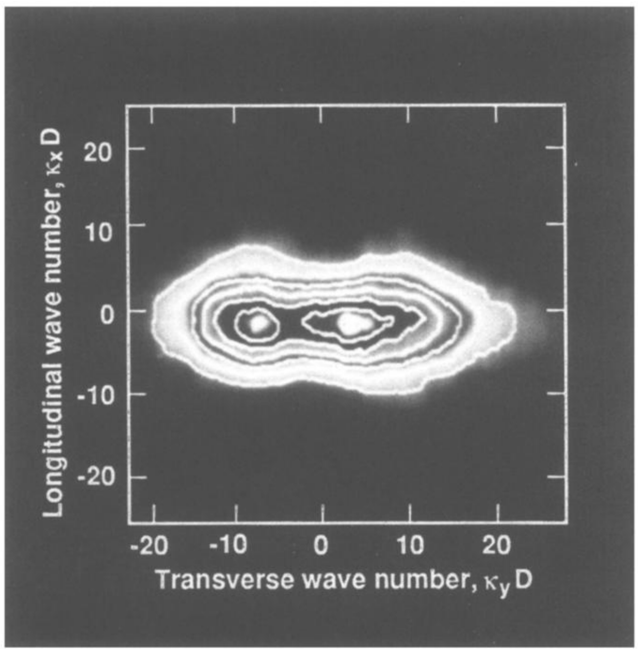

FIG. 1. Photograph and contours of 2D spectral density, $\Phi_\theta(\kappa_x, \kappa_y)$.