

PHYSICAL REVIEW LETTERS

VOLUME 65

10 SEPTEMBER 1990

NUMBER 11

Scattering for 1D Schrödinger Equation with Energy-Dependent Potentials and the Recovery of the Potential from the Reflection Coefficient

Tuncay Aktosun

Department of Mathematics, Southern Methodist University, Dallas, Texas 75275

Cornelis van der Mee

Department of Physics and Astronomy, Free University, Amsterdam, The Netherlands

(Received 16 April 1990)

We consider the 1D Schrödinger equation with a potential proportional to energy. When the spatial part of the potential is twice continuously differentiable, is less than 1 everywhere, and satisfies a certain integrability condition, we compute the scattering matrix. For the same class of potentials, under the further assumption that the potential is non-negative, we obtain the potential from one of the reflection coefficients.

PACS numbers: 03.65.Nk, 03.40.Kf, 03.80.+r, 43.20.+g

Consider the one-dimensional Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2} + k^2\psi(k,x) = k^2V(x)\psi(k,x), \quad (1)$$

where $x \in \mathbb{R}$ is the space coordinate, $k^2 \in \mathbb{R}$ is energy, and $k^2V(x)$ is the energy-dependent potential. For convenience we will call $V(x)$ the potential; $V(x)$ is assumed to decrease to zero sufficiently fast as $x \rightarrow \pm\infty$. Thus, there are two solutions of (1), which we will call the physical solutions ψ_l from the left and ψ_r from the right, which satisfy

$$\psi_l(k,x) = \begin{cases} T_l(k)e^{ikx} + o(1), & x \rightarrow \infty, \\ e^{ikx} + L(k)e^{-ikx} + o(1), & x \rightarrow -\infty, \end{cases}$$

$$\psi_r(k,x) = \begin{cases} e^{-ikx} + R(k)e^{ikx} + o(1), & x \rightarrow \infty, \\ T_r(k)e^{-ikx} + o(1), & x \rightarrow -\infty. \end{cases}$$

Here T_l and T_r are the transmission coefficients from the left and from the right, respectively, and L and R are the reflection coefficients from the left and from the right, respectively. The scattering matrix $S(k)$ is defined as

$$S(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}.$$

When $V(x)$ is real, we have $T_l = T_r$ and the common

value will be denoted by $T(k)$.

The Fourier transformation from the frequency (k) domain into the time (t) domain changes (1) into the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c(x)^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad (2)$$

where $c(x) = 1/\sqrt{1-V(x)}$ is the wave speed. Equation (2) describes the propagation of waves (e.g., sound waves or elastic waves) in a medium where the wave speed depends on position. To have a meaningful wave speed we assume that $V(x) < 1$ everywhere. We will also assume that $V(x)$ is twice continuously differentiable and that $V(x)$ and $G(x)$ are integrable, where $G(x)$ is defined as

$$G(x) = \frac{1}{4} \frac{V''(x)}{[1-V(x)]^{3/2}} + \frac{5}{16} \frac{V'(x)^2}{[1-V(x)]^{5/2}}. \quad (3)$$

These are all sufficient assumptions and may perhaps be relaxed.

The scattering problem for (1) consists of finding the scattering matrix when the potential is known; the inverse scattering problem is to recover the potential $V(x)$ when the scattering matrix, or equivalently one of the reflection coefficients, is known.

Equation (1) is related to the regular Schrödinger

equation

$$\frac{d^2\varphi(k,x)}{dx^2} + k^2\varphi(k,x) = V(x)\varphi(k,x). \quad (4)$$

Let us compare (1) and (4). Equation (4) is an eigenvalue problem for the Hamiltonian $-d^2/dx^2 + V(x)$, whereas (1) is not an eigenvalue problem; in (4) the solutions with the appropriate asymptotic behavior as $k \rightarrow \pm\infty$ and those with the appropriate asymptotic behavior as $x \rightarrow \pm\infty$ are related to each other in a simple manner, whereas for (1) this is not apparent. These are the main reasons why the scattering and inverse scattering problems for (1) are more difficult. We overcome these difficulties by finding two linearly independent solutions of (1) with the appropriate asymptotic behavior as $k \rightarrow \pm\infty$ and by showing how these two solutions are related to the physical solutions ψ_l and ψ_r .

There have been two methods to deal with the inverse problem for (1). The first method was proposed by Ware and Aki,¹ and it utilizes transforming (1) into the usual Schrödinger equation by using the travel-time coordinate

$$y = \int_0^x [1 - V(\xi)]^{1/2} d\xi$$

and the new wave function

$$\phi(k,y) = [1 - V(x)]^{1/4} \psi(k,x).$$

Then (1) is transformed into the Schrödinger equation

$$\frac{d^2\phi}{dy^2} + k^2\phi = Q(y)\phi,$$

where the new potential $Q(y)$ is related to the potential of (1) by

$$Q(y) = -\frac{5}{16} \frac{V'(x)^2}{[1 - V(x)]^3} - \frac{1}{4} \frac{V''(x)}{[1 - V(x)]^2}. \quad (5)$$

We use the prime to denote the derivative with respect to x . Then, Ware and Aki used the Marchenko method² to recover $Q(y)$ from the Fourier transform of the reflection coefficient. Then the potential $V(x)$ of (1) was assumed to be obtained from $Q(y)$ by inverting (5). However, the recovery of $V(x)$ from $Q(y)$ by inverting (5) presupposes the knowledge of $V(x)$, and thus the method of Ware and Aki does not give $V(x)$ when the scattering matrix is known. In their method Ware and Aki assumed that the potential has compact support.

The second method was proposed by Razavy,³ and it uses the spatial coordinate x rather than the travel-time coordinate y . Razavy's method is based on the iterative technique of Jost and Kohn⁴ and is more suited to find the potential approximately; in this method the potential is expressed as an infinite series where each term is an integral of some function of the reflection coefficient. However, the terms in the series become complicated even after the first term and the convergence is not assured.

We solve the scattering problem by computing $S(k)$ when $V(x)$ is given. The solution of the inverse problem given here is more straightforward in the sense that we

use the spatial coordinate rather than the travel-time coordinate and find $V(x)$ directly. We formulate the inverse scattering problem for (1) as a Riemann-Hilbert problem. Once the problem is posed as a Riemann-Hilbert problem, it can be solved by several methods such as the Marchenko method,^{2,5,6} and Gel'fand-Levitan method,^{6,7} the Wiener-Hopf factorization method,⁸ and the Muskhelishvili-Vekua method.^{9,10}

Using techniques similar to those used by Erdélyi,¹¹ we have the following result.

Theorem 1.—When $V(x)$ is twice continuously differentiable, $V(x)$ and $G(x)$ are integrable, and $V(x) < 1$, the Schrödinger equation (1) has two linearly independent solutions $Y_1(k,x)Z_1(k,x)$ and $Y_2(k,x)Z_2(k,x)$ such that

$$Y_1(k,x) = \frac{\exp[ik \int_0^x \sqrt{1 - V(\xi)} d\xi]}{[1 - V(x)]^{1/4}},$$

$$Y_2(k,x) = \frac{\exp[-ik \int_0^x \sqrt{1 - V(\xi)} d\xi]}{[1 - V(x)]^{1/4}},$$

and for each x , $Z_1(k,x)$ and $Z_2(k,x)$ are analytic in k in the upper half complex plane \mathbb{C}^+ , continuous in k for $k \in \mathbb{C}^+ \cup \mathbb{R}$, $Z_1(k,x) = 1 + O(1/k)$ and $Z_2(k,x) = 1 + O(1/k)$ as $k \rightarrow \infty$ in $\mathbb{C}^+ \cup \mathbb{R}$, $Z_1(k,+\infty) = 1$ and $Z_2(k,-\infty) = 1$, $Z_1'(k,+\infty) = 0$ and $Z_2'(k,-\infty) = 0$, and $Z_1(k,x)$ and $Z_2(k,x)$ satisfy the integral equations

$$Z_1(k,x) = 1 + \int_x^{+\infty} \mathcal{L}_1(k;x,\xi) Z_1(k,\xi) d\xi, \quad (6)$$

$$Z_2(k,x) = 1 - \int_{-\infty}^x \mathcal{L}_2(k;x,\xi) Z_2(k,\xi) d\xi, \quad (7)$$

where the kernels \mathcal{L}_1 and \mathcal{L}_2 are bounded in x and ξ and are given by

$$\mathcal{L}_1(k;x,\xi) = \frac{1}{2ik} \left[1 - \exp \left\{ 2ik \int_x^\xi \sqrt{1 - V(\eta)} d\eta \right\} \right] \times G(\xi),$$

$$\mathcal{L}_2(k;x,\xi) = -\frac{1}{2ik} \left[1 - \exp \left\{ -2ik \int_x^\xi \sqrt{1 - V(\eta)} d\eta \right\} \right] \times G(\xi).$$

Here $G(x)$ is the quantity given in (3).

The physical solutions of (1) can be written in terms of the functions given in Theorem 1 as

$$\psi_l(k,x) = T(k) \exp \left\{ ik \int_0^\infty [1 - \sqrt{1 - V(\eta)}] d\eta \right\} \times Y_1(k,x) Z_1(k,x), \quad (8)$$

$$\psi_r(k,x) = T(k) \exp \left\{ ik \int_{-\infty}^0 [1 - \sqrt{1 - V(\eta)}] d\eta \right\} \times Y_2(k,x) Z_2(k,x). \quad (9)$$

Note that when $V(x)$ is integrable, $1 - \sqrt{1 - V(x)}$ is also integrable because $|1 - \sqrt{1 - V}| = |V|/(1 + \sqrt{1 - V}) \leq |V|$.

It is known^{1,12} that the potential in (1) supports no bound states when $V(x) < 1$. The direct scattering prob-

lem is solved as follows. When the potential $V(x)$ is given, we obtain $G(x)$ from (3), $Z_1(k, x)$ from (6), and $Z_2(k, x)$ from (7). Then from the large- x asymptotic behavior of (8) and (9), the transmission and reflection coefficients are obtained as

$$\begin{aligned} \frac{1}{T(k)} &= \exp\left[ik \int_{-\infty}^{\infty} [1 - \sqrt{1 - V}] \right] \left[1 + \int_{-\infty}^{\infty} \frac{G(t)Z_1(k, t)}{2ik} dt \right], \\ \frac{1}{T(k)} &= \exp\left[ik \int_{-\infty}^{\infty} [1 - \sqrt{1 - V}] \right] \left[1 + \int_{-\infty}^{\infty} \frac{G(t)Z_2(k, t)}{2ik} dt \right], \\ \frac{L(k)}{T(k)} &= \exp\left[ik \int_0^{\infty} [1 - \sqrt{1 - V}] - ik \int_{-\infty}^0 [1 - \sqrt{1 - V}] \right] \\ &\quad \times \int_{-\infty}^{\infty} \frac{G(t)Z_1(k, t)}{2ik} \exp\left[2ikt - 2ik \int_0^t [1 - \sqrt{1 - V}] \right] dt, \\ \frac{R(k)}{T(k)} &= - \exp\left[ik \int_{-\infty}^0 [1 - \sqrt{1 - V}] - ik \int_0^{\infty} [1 - \sqrt{1 - V}] \right] \\ &\quad \times \int_{-\infty}^{\infty} \frac{G(t)Z_2(k, t)}{2ik} \exp\left[-2ikt + 2ik \int_0^t [1 - \sqrt{1 - V}] \right] dt. \end{aligned}$$

From the above expressions, as $|k| \rightarrow \infty$ we obtain

$$\begin{aligned} T(k) &= \exp\left[-ik \int_{-\infty}^{\infty} [1 - \sqrt{1 - V}] \right] \left[1 - \int_{-\infty}^{\infty} \frac{G(t)}{2ik} dt + O(1/k^2) \right], \quad k \in \mathbb{C}^+ \cup \mathbb{R}, \\ L(k) &= - \exp\left[-2ik \int_{-\infty}^0 [1 - \sqrt{1 - V}] \right] \int_{-\infty}^{\infty} \frac{G(t)}{2ik} \exp\left[2ikt - 2ik \int_0^t [1 - \sqrt{1 - V}] \right] dt + O(1/k^2), \quad k \in \mathbb{R}, \\ R(k) &= - \exp\left[-2ik \int_0^{\infty} [1 - \sqrt{1 - V}] \right] \int_{-\infty}^{\infty} \frac{G(t)}{2ik} \exp\left[-2ikt + 2ik \int_0^t [1 - \sqrt{1 - V}] \right] dt + O(1/k^2), \quad k \in \mathbb{R}. \end{aligned}$$

We also have

$$\begin{aligned} T(k) &= 1 + \frac{k}{2i} \int_{-\infty}^{\infty} dy e^{-iky} V(y) \psi_l(k, y) \\ &= 1 + \frac{k}{2i} \int_{-\infty}^{\infty} dy e^{-iky} V(y) \psi_r(k, y), \\ L(k) &= \frac{k}{2i} \int_{-\infty}^{\infty} dy e^{iky} V(y) \psi_l(k, y), \\ R(k) &= \frac{k}{2i} \int_{-\infty}^{\infty} dy e^{-iky} V(y) \psi_r(k, y), \end{aligned}$$

from which we obtain as $k \rightarrow 0$

$$\begin{aligned} T(k) &= 1 + \frac{k}{2i} \int_{-\infty}^{\infty} dy V(y) + O(k^2), \quad k \in \mathbb{C}^+ \cup \mathbb{R}, \\ L(k) &= \frac{k}{2i} \int_{-\infty}^{\infty} dy e^{2iky} V(y) + O(k^2), \quad k \in \mathbb{R}, \\ R(k) &= \frac{k}{2i} \int_{-\infty}^{\infty} dy e^{-2iky} V(y) + O(k^2), \quad k \in \mathbb{R}. \end{aligned}$$

The inverse scattering problem for (1), which consists of the determination of $V(x)$ from the reflection coefficient, is an important problem. It is equivalent to the determination of the wave speed $c(x)$ from one of the reflection coefficients and has many important applications in acoustic imaging, nondestructive testing, and various fields of geophysics such as seismology. For the class of potentials mentioned in Theorem 1 and under the further assumption $V(x) \geq 0$, we show that $V(x)$ can be recovered uniquely from one of the reflection

coefficients. We will first formulate the inverse scattering problem for (1) as a Riemann-Hilbert boundary-value problem.^{10,13}

Since k appears as k^2 in (1), $\psi_l(-k, x)$ and $\psi_r(-k, x)$ are also solutions of (1) whenever $\psi_l(k, x)$ and $\psi_r(k, x)$ are solutions. The solution vectors

$$\psi(-k, x) = \begin{bmatrix} \psi_l(-k, x) \\ \psi_r(-k, x) \end{bmatrix}$$

and

$$\psi(k, x) = \begin{bmatrix} \psi_l(k, x) \\ \psi_r(k, x) \end{bmatrix}$$

are related to each other as

$$\psi(-k, x) = S(k)^{-1} q \psi(k, x), \quad k \in \mathbb{R}, \quad (10)$$

where $q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $S(k)^{-1}$ denotes the matrix inverse of $S(k)$. Let us define

$$m_l(k, x) = \frac{1}{T(k)} e^{-ikx} \psi_l(k, x), \quad (11)$$

$$m_r(k, x) = \frac{1}{T(k)} e^{ikx} \psi_r(k, x).$$

Letting

$$m(k, x) = \begin{bmatrix} m_l(k, x) \\ m_r(k, x) \end{bmatrix},$$

we can write (10) as

$$m(-k, x) = \Lambda(k, x) q m(k, x), \quad k \in \mathbb{R}, \quad (12)$$

where

$$\Lambda(k, x) = \begin{bmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{-2ikx} & T(k) \end{bmatrix}.$$

Under the transformation $k \rightarrow -1/k$, \mathbb{C}^+ and $\mathbb{R} \cup \{\infty\}$ are mapped onto themselves in a one-to-one manner. Let us use the notation $\tilde{F}(k) = F(-1/k)$ throughout. Then we can write (12) as

$$\tilde{m}(-k, x) = \tilde{\Lambda}(k, x) q \tilde{m}(k, x), \quad k \in \mathbb{R}. \quad (13)$$

It can be shown that $\tilde{m}(k, x)$ is continuous in $k \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$, has an analytic extension in k to \mathbb{C}^+ for each x , and $\tilde{m}(k, x) = \hat{1} + O(1/k)$ as $k \rightarrow \infty$ in $\mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$, where we define $\hat{1} = [1]$. Similarly, $\tilde{m}(-k, x)$ has an analytic extension in k to the lower half complex plane \mathbb{C}^- for each x , is continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$, and $\tilde{m}(-k, x) = \hat{1} + O(1/k)$ and $k \rightarrow \infty$ in $\mathbb{C}^- \cup \mathbb{R}$. Hence, (13) is a Riemann-Hilbert problem. We will

$$B_l(x, y) = g_l(x, y) + \int_0^\infty dz g_l(x, y + z) B_l(x, z), \quad y > 0, \quad (16)$$

$$B_r(x, y) = g_r(x, y) + \int_0^\infty dz g_r(x, y + z) B_r(x, z), \quad y > 0, \quad (17)$$

$$B_l(x, y) + g_r(x, -y) + u(-y) + \int_0^\infty dz g_r(x, -y + z) B_r(x, z) + \int_0^y dz u(-y + z) B_l(x, z) = 0, \quad y > 0, \quad (18)$$

$$B_r(x, y) + g_l(x, -y) + u(-y) + \int_0^\infty dz g_l(x, -y + z) B_l(x, z) + \int_0^y dz u(-y + z) B_r(x, z) = 0, \quad y > 0, \quad (19)$$

where (16) and (17) are the uncoupled Marchenko equations and (18) and (19) are the coupled Wiener-Hopf equations. Let us write (16) and (17) in operator form as

$$B = g + \mathcal{G}B, \quad (20)$$

where $B \in L^2(0, \infty)$ is the unknown function and $g \in L^2(0, \infty)$ is given. Here $L^2(0, \infty)$ is the Hilbert space of square-integrable functions on $(0, \infty)$. We then have the following result concerning the solvability of (20).

Theorem 2.—If the potential $V(x)$ is twice continuously differentiable, $V(x)$ and $G(x)$ are integrable, and $0 \leq V(x) < 1$, then the operator \mathcal{G} in (20) is self-adjoint on $L^2(0, \infty)$ and its operator norm satisfies $\|\mathcal{G}\| < 1$. Thus the Marchenko integral equations (16) and (17) are uniquely solvable.

Since $\|\mathcal{G}\| < 1$, (20) can be solved by using the Neumann expansion $B = \sum_{n=0}^\infty \mathcal{G}^n g$. Once $B_l(x, y)$ and $B_r(x, y)$ are obtained from the Marchenko equation, by using $B_l(x, y) = B_r(x, y) = 0$ for $y < 0$ and using the inverse Fourier transform on (14) and (15), we obtain $\tilde{m}_l(k, x)$ and $\tilde{m}_r(k, x)$. Then, we have $m_l(k, x) = \tilde{m}_l(-1/k, x)$ and $m_r(k, x) = \tilde{m}_r(-1/k, x)$. The functions $\tilde{m}_l(k, x)$ and $\tilde{m}_r(k, x)$ obtained this way solve the Riemann-Hilbert problem (13) if and only if $B_l(x, y)$ and $B_r(x, y)$ are also solutions of the Wiener-Hopf equations (18) and (19). Then, the potential $V(x)$ can be

solve (13) by the Marchenko method. Let

$$u(y) = \int_{-\infty}^\infty \frac{dk}{2\pi} [\tilde{T}(k) - 1] e^{iky},$$

$$B_l(x, y) = \int_{-\infty}^\infty \frac{dk}{2\pi} [\tilde{m}_l(k, x) - 1] e^{-iky}, \quad (14)$$

$$g_l(x, y) = - \int_{-\infty}^\infty \frac{dk}{2\pi} R(-1/k) e^{-2ix/k} e^{iky},$$

$$B_r(x, y) = \int_{-\infty}^\infty \frac{dk}{2\pi} [\tilde{m}_r(k, x) - 1] e^{-iky}, \quad (15)$$

$$g_r(x, y) = - \int_{-\infty}^\infty \frac{dk}{2\pi} L(-1/k) e^{2ix/k} e^{iky}.$$

From the analyticity and asymptotic properties of $\tilde{m}(k, x)$ for $k \in \mathbb{C}^+$ mentioned above, we have $B_l(x, y) = B_r(x, y) = 0$ for $y < 0$. Then the Riemann-Hilbert problem (13), upon Fourier transformation, is equivalent to the four equations:

recovered from (1) as

$$V(x) = \frac{m_l''(k, x) + 2ikm_l'(k, x)}{k^2 m_l(k, x)} = \frac{m_r''(k, x) + 2ikm_r'(k, x)}{k^2 m_r(k, x)}, \quad (21)$$

provided the middle and the right-hand side are equal and independent of k . A different way to obtain $V(x)$ is to exploit the large- k asymptotic behavior of $m_l(k, x)$ and $m_r(k, x)$. From (8), (9), and (11), we have

$$\lim_{k \rightarrow \pm\infty} \frac{-i \ln m_l(k, x)}{k} = \int_x^\infty [1 - \sqrt{1 - V(\xi)}] d\xi, \quad (22)$$

$$\lim_{k \rightarrow \pm\infty} \frac{-i \ln m_r(k, x)}{k} = \int_{-\infty}^x [1 - \sqrt{1 - V(\xi)}] d\xi. \quad (23)$$

Hence, instead of using (21), the potential can also be obtained from (22) or (23) through differentiation, provided the same result is obtained from (22) and (23); the condition that (22) and (23) lead to the same potential $V(x)$ is the analog of the 1D version of Newton's miracle condition.⁶

All the proofs and mathematical details of the results outlined in this paper will be published elsewhere.¹⁴

The authors are indebted to Roger Newton for his help. The research leading to this article was supported in part by the National Science Foundation under Grants No. DMS 8823102 and No. DMS 9001903.

- ¹J. A. Ware and K. Aki, *J. Acoust. Soc. Am.* **45**, 911 (1969).
²L. D. Faddeev, *Am. Math. Soc. Transl.* **2**, 139 (1964).
³M. Razavy, *J. Acoust. Soc. Am.* **58**, 956 (1975).
⁴R. Jost and W. Kohn, *Phys. Rev.* **87**, 977 (1952).
⁵Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963).
⁶R. G. Newton, *J. Math. Phys.* **21**, 493 (1980).
⁷I. M. Gel'fand and B. M. Levitan, *Am. Math. Soc. Transl.* **1**, 253 (1955).
⁸T. Aktosun and C. van der Mee, *J. Math. Phys.* (to be published).
⁹N. I. Muskhelishvili and N. P. Vekua, *Trudy Tbilissk. Mat. Inst.* **12**, 1 (1943) (in Russian).
¹⁰N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953).
¹¹A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956).
¹²P. Deift and E. Trubowitz, *Commun. Pure Appl. Math.* **32**, 121 (1979).
¹³F. D. Gakhov, *Boundary Value Problems* (Pergamon, Oxford, 1966).
¹⁴T. Aktosun and C. van der Mee, *J. Math. Phys.* (to be published).