Experimental Study of Critical-Mass Fluctuations in an Evolving Sandpile

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(Received 9 May 1990)

We have carried out an experiment in which sandpiles are built up to a critical size and then perturbed by the dropping of individual grains of sand onto the pile. After each grain is added, the size of the resulting avalanche, if any, is recorded. For sufficiently small sandpiles, the observed mass fluctuations are scale invariant and the probability distribution of avalanches shows finite-size scaling. This demonstrates that real, finite-size sandpiles may be described by models of self-organized criticality. However, we also find that this description breaks down in the limit of large sandpiles.

PACS numbers: 64.60.Fr, 05.40.+j, 05.70.Jk, 46.10.+z

Recent theoretical investigations have shown that when certain spatially extended nonequilibrium systems are driven, they naturally evolve into a critical state.[1-4] This state is barely stable and, when it is perturbed, the resulting relaxation processes are scale invariant. That is, they are characterized by an infinite correlation length, and so occur on all length and time scales up to limits determined by the finite size of the system. The occurrence of such critical states in nonequilibrium systems is spontaneous; it does not require, as in equilibrium systems, the tuning of experimentally adjustable parameters to particular values or critical points. The robust nature of this "self-organized criticality" had led Bak, Tang, and Wiesenfeld[1] to suggest that it may account for the ubiquitous presence in nature of scale-invariant phenomena such as $1/f$ noise[5] and fractal structures.[6]

One system which has been studied extensively as a paradigm for self-organized criticality is the theoretical sandpile.[1-4,7,8] Several cellular-automaton models of sandpiles have been shown both numerically and analytically[8] to exhibit scale invariance under generic conditions. As theoretical grains are dropped onto a pile which has reached a steady state, the resulting distributions of both avalanche sizes and lifetimes typically exhibit scale-invariant behavior, i.e., fall off as powers of avalanche size and lifetime, respectively. Experimentally, power-law dependences of this type have been reported for "avalanches" of domain-wall boundaries of metastable cellular patterns in magnetic garnet films.[9] However, in an experimental study of sand at the angle of repose, Jaeger, Liu, and Nagel[10] reported avalanche distributions in both size and lifetime which were sharply peaked at nonzero values, in contrast to the theoretical models.

We present here the results of an experimental study of the dynamics of an evolving sandpile. The sandpile is built up to a steady state and then subsequently perturbed by the addition of single grains of sand. After each grain is added the size of the resulting avalanche, if any, is recorded. Our main results follow: Sufficiently small sandpiles show broad, scale-invariant distributions of avalanche sizes. Moreover, the distributions exhibit finite-size scaling in the linear size L of the pile. This suggests that these sandpiles are in a self-organized critical state. However, for larger sandpiles we find that the distribution of avalanches becomes sharply peaked, and the scale invariant description of the system breaks down. This is discussed below.

Our experiment differs in two main ways from those of Ref. 10. First, our sandpiles are built on circular disks and, thus, the flow of sand down the pile is intrinsically two dimensional. In the experimental geometries employed in Ref. 10 the sand was inclined along only one direction. Second, by adding sand to the pile one grain at a time, and always waiting for avalanches to subside before dropping another grain, we are able to parallel more closely cellular-automaton models known to exhibit self-organized criticality.

Our experimental apparatus is illustrated in Fig. 1. The sand is dispensed through a funnel which was formed by fusing a 9-in.-long 0.080-in.-i.d. capillary tube to the end of a 250-ml leveling bulb. This funnel is filled with sand and rotated about its axis by a computer-controlled dc motor at approximately 1 rev/s. The capillary tube is angled slightly downward, and the rotation results in the formation of a row of sand grains which travel single file through the tube. When a grain falls out of the tube it lands near the top of a sandpile which is built on a circular disk, which, in turn, is supported on the weighing pan of a Mettler AT250 analytic balance. The balance is interfaced to an IBM PC which monitors the mass of the sandpile. When a change in mass comparable to a grain of sand is detected, the computer stops

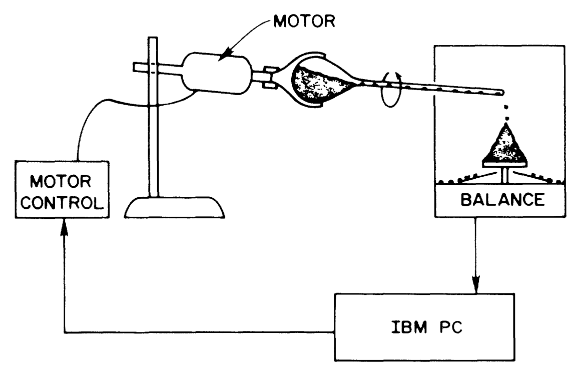

FIG. 1. Schematic illustration of the experimental apparatus.

the rotation of the funnel (and thus the flow of sand) until any avalanches have occurred and the mass of the pile has restabilized. The dropping process is then resumed. Any sand which falls off the pile lands on a metal skirt and is no longer included in the mass measured by the balance. In this fashion, we obtain the mass of the sandpile as a discrete function of the number of grains dropped onto it; irregularities in the dropping rate, as well as variations in the durations of the avalanches, are effectively removed from the data. The average time between dropping events is approximately 10 s and the lateral precession of the end of the capillary tube over the sandpile is approximately 0.1 in. For most of the experiments the sand grains fell 3–4 in. between the end of the capillary tube and the top of the sandpile.

The data shown below were taken using aluminum-oxide particles sieved between 20- and 25-mesh/in. screens. The average mass of a particle was 0.0006 g. Sandpiles with base diameters of 0.38, 0.75, 1.5, and 3.0 in. were studied. Comparable results were obtained using similarly sieved beach sand with 0.75- and 1.5-in.-diam bases.

In Fig. 2(a) we plot the fluctuations in the mass of a sandpile with a 1.5-in.-diam base over 30000 dropping events. In Fig. 2(b) we show a 15× magnification of the small boxed region of Fig. 2(a). In Fig. 2(c) we magnify 20× the small boxed region of Fig. 2(b). On this last scale, changes in mass corresponding to single grains of sand are observable. It is clear from these plots that the mass of this sandpile exhibits fluctuations over periods ranging from one to several thousand dropping events, with avalanches ranging between one and several hundred grains of sand.

To quantify these fluctuations, we plot in Fig. 3(a) the probability density $P(M)$ of an avalanche of mass M for sandpiles with base diameters of 0.38, 0.75, and 1.5 in. For all of these pan sizes, the probability of an avalanche falls off monotonically with increasing avalanche size. For the 1.5-in.-diam sandpile, this falloff is approximately a power law, $P(M) \sim M^{-2.5}$, for avalanches between 0.002 and 0.05 g (i.e., between 3 and 80 grains). The falloff in $P(M)$ is not exponential for any of the sandpiles, nor is $P(M)$ sharply peaked at a particular avalanche size. With increasing base diameter, the probability of large avalanches increases while that of the small avalanches decreases. If this system does indeed exhibit scale-invariant fluctuations similar to those observed at the critical point of a second-order phase transition, one would expect the distribution of avalanches to

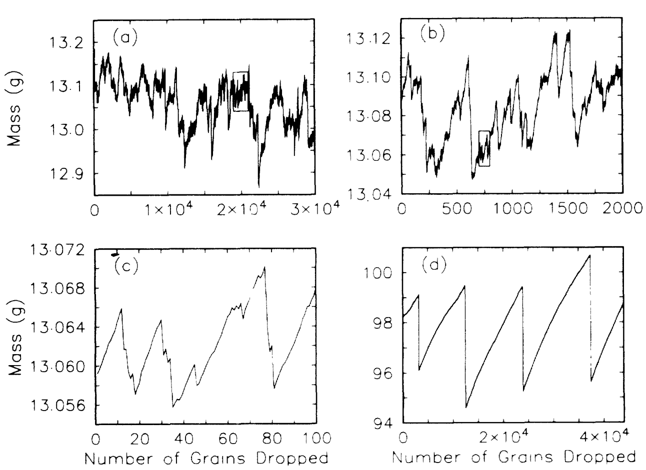

FIG. 2. (a) Mass of the 1.5-in.-diam sandpile as a function of the number of grains dropped onto it. (b) Magnification of the boxed region of (a) between 19000 and 21000 grains dropped. (c) Magnification of the boxed region of (b) between 700 and 800 grains dropped. (d) Mass of the 3.0-in.-diam sandpile as a function of the number of grains dropped onto it. The fine-scale mass fluctuations observed for the 1.5-in.-diam sandpile are no longer present.

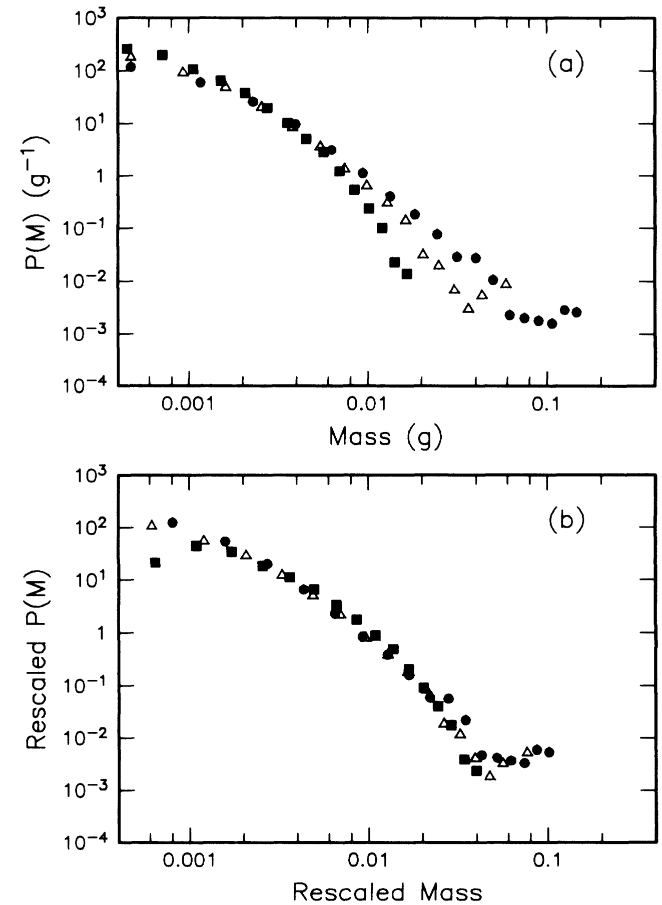

FIG. 3. (a) Probability distribution of avalanches $P(M)$ as a function of avalanche mass M for sandpiles with base diameters of 1.5 (●), 0.75 (△), and 0.38 (■) in. Units of probability density $P(M)$ are g^{-1}. For each base diameter, the range of avalanche sizes ΔM included in a given data point $P(M)$ increases quadratically with mass. (b) The probability distributions from (a) rescaled by Eq. (1) with $\beta = 2\nu = 1.8$. Mass and probability density are rescaled to $ML^{-0.9}$ and $P(M)L^{1.8}$, respectively, where L is the base diameter in inches.

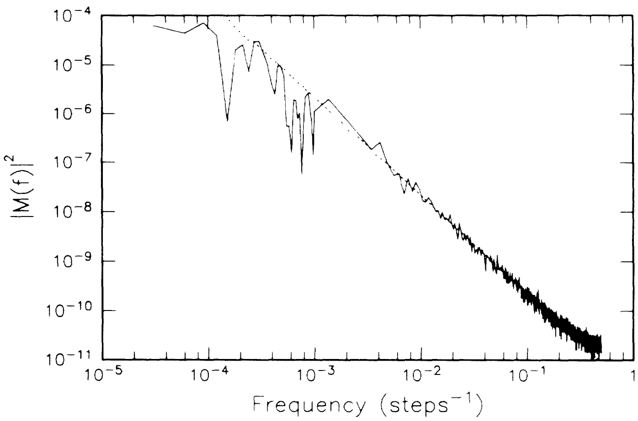

FIG. 4. Power spectrum of the mass fluctuations of the 1.5-in.-diam sandpile. The dashed line shows a $1/f^2$ power spectrum for comparison. The spacing between data points of the spectrum is 6.1×10^{-5} step^{-1}, with the data at frequencies above 0.002 step^{-1} smoothed by an averaging over 0.0014-step^{-1} intervals. The time unit (steps) is defined as the interval between grains of sand being dropped on the sandpile (see text).

show finite-size scaling of the form[1,3]

$$P(M,L)=(1/L^{\beta})g(M/L^{\nu}), \tag{1}$$

where $P(M,L)$ is the probability of an avalanche of mass M for a sandpile of base diameter L, and g is a universal function. Further, the requirement that for each grain added an average of one must fall off results in the constraint[3,11] $\beta=2\nu$. We rescale the data shown in Fig. 3(a) using Eq. (1), finding that for $\beta=2\nu=1.8$ the rescaled avalanche distributions for the different size sandpiles lie almost exactly on the same universal curve $g(M/L^{\nu})$ [Fig. 3(b)]. This indicates that the data do indeed show finite-size scaling. We note that cellular-automaton models for 2D sandpiles[3] were also reported to scale according to Eq. (1). The data in Figs. 2(a)–2(c) and 3 demonstrate that the sandpiles shown are in a self-organized critical state.

The mass power spectrum of the time series for the 1.5-in.-diam sandpile is shown in Fig. 4. At frequencies above $f_0=0.0003$ step^{-1}, this spectrum falls off as $1/f^2$. This is consistent with the power spectrum of a weighted random walk. That is, if the net mass change $\eta(t)$ resulting from an added grain is on average zero [i.e., $\langle\eta(t)\rangle=0$] while the correlation function $\langle\eta(t)\eta(t')\rangle=(\eta_0)^2\delta(t-t')$ for some η_0 not equal to zero, it is easy to show that one obtains a $1/f^2$ power spectrum. Thus, the $1/f^2$ spectrum of Fig. 4 suggests that the sizes of the different avalanches in our data have at most short-range correlations in time. At frequencies below f_0, the power spectrum levels off. This is expected because the mass of the sandpile, and thus the random walk, is bounded by an upper limit M_0. Increasing the base diameter of the sandpile, we observe both an increase in M_0 and a corresponding shift downward in f_0.

The unit of time used in computing the above power spectrum is the time between dropping events. That is, regardless of the actual time elapsed between two grains of sand being dropped, the time interval is recorded as a single time step. As a result, the actual durations of the different avalanches have no effect on this power spectrum—all of the avalanches are effectively instantaneous. However, because even the largest avalanches occur on a time scale negligible compared to the time between dropping events, this definition does not affect the power spectrum over the frequency range measured. This is a consequence of the finite size of our system; if the sandpile were arbitrarily large (and still remained in a self-organized critical state), there would presumably be avalanches occurring on arbitrarily long time scales, and so treating the avalanches as instantaneous would strongly affect the spectrum. However, even in the limit of an infinite system and perfect experimental time resolution, it is not clear that the mass power spectrum of a self-organized critical sandpile would show $1/f^{\alpha}$ noise with $\alpha\sim1$. The falloff of the power spectrum depends importantly on the temporal structure of the individual avalanches, the distribution of avalanche durations, and the manner in which different avalanches are superposed. Recent numerical simulations of an automaton-model sandpile, which certainly shows self-organized criticality, found it to exhibit a $1/f^2$ power spectrum.[7]

In Fig. 2(d) we plot $M(t)$ for a 3-in.-diam base sandpile with a distance of 0.25 in. between the capillary tube and the top of the sandpile. It is quite clear that this system behaves in a fundamentally different manner from the others described above; the small fluctuations in mass have disappeared and almost all of the mass flow off the sandpile occurs through large, regularly spaced avalanches. This behavior is essentially that of a relaxational oscillator, and is similar to that reported in Ref. 10. There are several qualitative differences between the 3-in.-diam sandpile and the smaller ones which could account for this crossover. For the small sandpiles, almost all of the disturbances which propagate more than a few grain diameters result in some net flow off the pile. In contrast, on the 3-in. pile there are many local avalanches which propagate for distances of an inch or more, yet stabilize without any grains falling off the pile. Additionally, most of the sand which flows off the smaller piles appears to originate near the surfaces of these piles, whereas the large avalanches observed for the 3-in.-diam pile result from the flow of sand further below the surface. After an avalanche, one side of this pile is noticeably concave. The presence or absence of self-organized criticality may therefore be related to a damping length scale which we do not fully understand.

Another important experimental difference between this 3-in. sandpile and the smaller ones is that the distance between the dropper and the top of the 3-in. pile is much smaller. As a result, less kinetic energy is added to the system with each grain. This presumably enhances the role of static friction in the behavior of the pile, which can build up to a limit determined by the coefficient of static friction, and then relax back through the large avalanches to an angle of repose determined by the coefficient of sliding friction, which is smaller. To better determine the relationship between the height from which the grains are dropped and the resulting sandslide dynamics, we have collected data at various dropper heights for the 1.5- and 3-in. sandpiles. We find that while raising the dropper to a height 4 in. above the top of the 3-in. sandpile does introduce some small avalanches, the net mass flow continues to be dominated by the large (~ 5 g) avalanches. Further, setting the dropper height at 0.25 in. above the 1.5-in.-diam sandpile does increase the number of large avalanches, but a significant number of small mass fluctuations remain. We have not yet been able to observe cleanly a crossover from self-organized critical behavior to relaxational oscillations by simply decreasing the height of the dropper over a single sandpile. Still, it seems reasonably clear that in a series of measurements wherein the diameter of the sandpile is increased while the height difference between the dropper and the top of the pile is held fixed, the behavior of the system will exhibit such a crossover. Additionally, we have increased the lateral region over which the grains are dropped onto the 3-in. sandpile by increasing the lateral precession of the dropper to 0.3 in. We find that this increase has no apparent effect on the system. Further experiments to determine the exact nature of the transition between the regimes of self-organized criticality and relaxation oscillations are currently in progress.

In conclusion, we have observed self-organized criticality in the dynamics of several different size sandpiles. The mass fluctuations in small sandpiles show critical finite-size scaling similar to that associated with second-order phase transitions. However, the largest sandpile studied showed relaxational oscillations, rather than scale invariance. Thus, our data suggest, but do not prove, that the occurrence of self-organized criticality in the present experimental geometry is a finite-size effect. The size at which the system crosses over to relaxation oscillations depends on experimental parameters such as the distance between the dropper and the top of the sandpile. It is hoped that further studies will clarify the precise nature of this crossover.

We wish to thank Per Bak for helpful discussions.

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[11]The measured mass change M is the difference in the mass of the sandpile before and after a grain of sand has been dropped onto it. Following the definition of avalanche size in Ref. 3, a mass change of M actually corresponds to an avalanche of size $M+M_0$, where M_0 is the mass of a single grain. Thus, strictly speaking, the scaling relation of Eq. (1), as well as the steady-state constraint $\beta = 2\nu$, applies to $P(M+M_0,L)$. However, in the limit $M \gg M_0$ (where finite-size scaling is applicable), these relations should be valid for $P(M,L)$ as well.