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Reductions of Self-Dual Yang-Mills Fields and Classical Systems

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One-dimensional reductions of the self-dual Yang-Mills equations yield various classical systems depending on the choice of the Lie algebra associated with the self-dual fields. Included are the Euler-Arnold equations for rigid bodies in n dimensions, the Kovalevskaya top, and a generalization of the Nahm equation which is related to a classical third-order differential equation possessing a movable natural boundary in the complex plane.

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One of the richest "exactly solvable" systems is the self-dual Yang-Mills (SDYM) equations. These equations arise in the study of field theory[1] and relativity[2] and have wide classes of interesting solutions.[3] Recently it has been shown that these equations admit reductions to many well-known classical systems including the "soliton" equations in one space and one time $(1+1)$ dimension.[4,5]

In this Letter we study the reductions of SDYM equations in one $(0+1)$ dimension. We find as special cases the Euler-Arnold equations for rigid bodies, the Kovalevskaya top, and a generalization of the Nahm equation that we refer to as the "Chazy" top. The latter system can be transformed to a classical differential equation studied by Chazy.[6] The reduction of soliton systems generally possesses the so-called Painlevé property[7] (i.e., solutions have only movable poles in the complex plane). In fact, it was originally Kovalevskaya[8] who employed local singularity analysis to uncover the cases in which the equations of a rigid body in a gravitational field and with one point fixed could be integrated exactly. However, Chazy's equation possesses a natural boundary in the complex plane and as such differs from the other reductions of the SDYM equations described above which are solved in terms of Riemann theta functions.[9]

Define the YM field as

$$F_{ab}=\partial_a\gamma_b-\partial_b\gamma_a-[\gamma_a,\gamma_b]\,, \tag{1}$$

where $\partial_a=\partial/\partial x^a$, x^a being the coordinates in Euclidian space $\mathbf{E}^4$. The γ_a's are the YM potentials and take values in some Lie algebra with $[\,,\,]$ in (1) being the associated Lie bracket. In terms of standard null coordinates $\alpha=t+iz$, $\bar\alpha=t-iz$, $\beta=x+iy$, and $\bar\beta=x-iy$, the SDYM field equations are given by

$$F_{\alpha\beta}=F_{\bar\alpha\bar\beta}=F_{\alpha\bar\alpha}+F_{\beta\bar\beta}=0\,. \tag{2}$$

This is a system of three partial differential equations involving the γ_a's and can also be obtained as the compatibility condition of the following Lax pair:

$$\begin{aligned}(\partial_\alpha+\zeta\partial_{\bar\beta})\Theta&=(\gamma_\alpha+\zeta\gamma_{\bar\beta})\Theta\,,\\(\partial_\beta+\zeta\partial_{\bar\alpha})\Theta&=(\gamma_\beta+\zeta\gamma_{\bar\alpha})\Theta\,,\end{aligned} \tag{3}$$

where ζ is a complex parameter often referred to as the spectral parameter. The compatibility condition is expressed as a polynomial in ζ,

$$(\partial_\alpha+\zeta\partial_{\bar\beta})(\gamma_\alpha+\zeta\gamma_{\bar\beta})-(\partial_\beta+\zeta\partial_{\bar\alpha})(\gamma_\beta+\zeta\gamma_{\bar\alpha})=[(\gamma_\alpha+\zeta\gamma_{\bar\beta}),(\gamma_\beta+\zeta\gamma_{\bar\alpha})]\,. \tag{4}$$

Equating independent powers of ζ (ζ^n, $n=0,1,2$) yields (2).

Reductions are obtained by allowing the γ_a's to depend on only one independent variable and then choosing an appropriate gauge group for the field variables. Our motivation is as follows: (1) By letting the field variables depend on only the time coordinate the field equations are greatly simplified while maintaining physical content. (2) The reduced phase space of a Hamiltonian system admitting a group of symmetries can be described by the cotangent bundle of a Lie algebra with a canonical symplectic structure. It is therefore convenient to construct Lie-algebraic embeddings of the phase space and study the flows in the resulting symmetric spaces.[9,10] Our approach is to consider the phase-space variables as the YM field variables with the underlying gauge group as the symmetry group of the dynamical system and the SDYM equations describing the resulting phase flow. Examples follow.

Case I: The Euler-Arnold top.— Let $\gamma_a=\gamma_a(a)$. Then (4) can be written in the Lax form as

$$\partial_a \mathcal{L}_{\rm I}=[\mathcal{L}_{\rm I},\mathcal{M}_{\rm I}]\,, \qquad (5)$$

where $\mathcal{L}_{\rm I}=\gamma_\beta+\zeta\gamma_{\bar a}$ and $\mathcal{M}_{\rm I}=-(\gamma_a+\zeta\gamma_{\bar\beta})$.

The Lax form is intimately related to the integrability of the system since the conserved quantities are incorporated naturally. One can check from (5) that $\partial a\,{\rm Tr}(\mathcal{L}_{\rm I})^n=0$, $n=1,2\ldots$, which when expanded in a power series in the parameter ζ yields the constants of motion. These are the reduced versions of the infinitely many conserved currents associated with SDYM equations[1] which can also be obtained from the Lax pair (3).

The reduction is effected by allowing the γ_a's to take values in the Lie algebra su(n). In particular, we take $\gamma_{\bar a}={\rm diag}(\bar a_i)$ and $\gamma_{\bar\beta}={\rm diag}(\bar\beta_i)$, $i=1\ldots n$, to be in the Cartan subalgebra of su(n). Furthermore $\gamma_a=[\alpha_{ij}]$ and $\gamma_\beta=[\beta_{ij}]$ are chosen to be in the subalgebra so(n). Then from (5) we obtain after some manipulations[4]

$$\partial_a\beta_{ij}=\sum_k[1/(\bar\beta_i+\bar\beta_k)-1/(\bar\beta_j+\bar\beta_k)]\beta_{ik}\beta_{kj}\,, \qquad (6)$$

where we have set $\bar a_i=(\bar\beta_i)^2$. Equation (6) is the Euler-Arnold[10] equation for free motion of an n-dimensional rigid body about a fixed point. Specifically when $n=3$, taking $\beta_{ij}=L_k$, $\bar\beta_i+\bar\beta_j=I_k$, one recovers from (6) the Euler equation for a free, spinning top about a fixed point:

$$\partial_a L_i=L_jL_k(I_k-I_j)/I_jI_k,\quad i\neq j\neq k \text{ and cyclic}.$$

The L_i's are the usual angular momentum components in the body frame, the I_i's are the principal moments of inertia, and a is the complex "time" variable. The nontrivial constants of motion are obtained from (a) the coefficient of ζ^0 in ${\rm Tr}(\mathcal{L}_{\rm I})^2={\rm Tr}(\gamma_\beta)^2=\sum_i L_i^2$ and (b) the coefficient of ζ in ${\rm Tr}(\mathcal{L}_{\rm I})^3=3{\rm Tr}(\gamma_{\bar a}\gamma_\beta)^2$. The first yields the square of the total angular momentum; the Hamiltonian is then obtained from (b).

Case II: The Kovalevskaya top and its generalizations.— In this reduction we take $\gamma_a=\gamma_a(t)$ and in the so(p,q) algebra, Eq. (4) can be written in the Lax form as

$$\partial_t\mathcal{L}_{\rm II}=[\mathcal{L}_{\rm II},\mathcal{M}_{\rm II}]\,, \qquad (7)$$

where $\mathcal{L}_{\rm II}=\gamma_\beta-2i\zeta\gamma_z-\zeta^2\gamma_{\bar\beta}$ and $\mathcal{M}_{\rm II}=-(\gamma_t+i\gamma_z+\zeta\gamma_{\bar\beta})$. γ_z and $\gamma_t+i\gamma_z$ are chosen so that they lie in the maximal subalgebra $K={\rm so}(p)\oplus{\rm so}(q)$, whereas γ_β and $\gamma_{\bar\beta}$ are in the orthogonal complement to K. The classical Kovalevskaya system is obtained by considering the algebra so(3,1). The γ_a's are represented by the following 4×4 matrices expressed in a 2×2 block form:

$$\gamma_\beta=\begin{bmatrix}0_{3\times3} & V_{3\times1}\\ V^T_{1\times3} & 0_{1\times1}\end{bmatrix},\quad \gamma_z=\begin{bmatrix}L_{3\times3} & 0_{3\times1}\\ 0^T_{1\times3} & 0_{1\times1}\end{bmatrix},$$

$$\gamma_{\bar\beta}=\begin{bmatrix}0_{3\times3} & C_{3\times1}\\ C^T_{1\times3} & 0_{1\times1}\end{bmatrix},\quad \gamma_t+i\gamma_z=\begin{bmatrix}\omega_{3\times3} & 0_{3\times1}\\ 0^T_{1\times3} & 0_{1\times1}\end{bmatrix},$$

with $V^T_{1\times3}=(g_1,g_2,g_3)$ the field vector, $C^T_{1\times3}=(c_1,c_2,c_3)$ the center-of-mass vector, $L_{3\times3}=[L_{ij}]$, $L_{ij}=\sum_k\varepsilon_{ijk}L_k$, ε_{ijk} the totally skew symmetric tensor with $\varepsilon_{123}=1$, and L_k the angular momentum components. $\omega_{3\times3}$ has the same form as $L_{3\times3}$ with L_i replaced by ω_i (angular velocity); $L_i=I_i\omega_i$, I_i being the principal moments of inertia. Equation (7) yields the equation of a "heavy top" rotating about a fixed point in the moving frame:

$$\partial_t\mathbf{L}=\mathbf{L}\times\boldsymbol{\omega}+\mathbf{g}\times\mathbf{c},\quad \partial_t\mathbf{g}=\mathbf{g}\times\boldsymbol{\omega}.$$

$\mathbf{L}=(L_1,L_2,L_3)$ plus conditions on the I_i's depending on the choice for the center-of-mass vector $\mathbf{c}$ (essentially those of Kovalevskaya). For so(p,q) one can obtain from (7) generalizations of the Kovalavskaya case.[9,11] As in case I, the integrals of motion for the so(p,q) systems are obtained by expanding ${\rm Tr}(\mathcal{L}_{\rm II})^n$, $n=1,2\ldots$, as a power series in ζ. In particular, the coefficient of ζ^2 in ${\rm Tr}(\mathcal{L}_{\rm II})^2$ yields the Hamiltonian; the remaining nontrivial integrals of motion are obtained from traces of higher exponents of $\mathcal{L}_{\rm II}$.

Case III: The Nahm and Chazy equations.— We take $\gamma_a=\gamma_a(t)$ as in the previous case and $\gamma_t=0$ by utilizing the gauge freedom of the SDYM field equations (2). Then from (2) or (4) we find that the remaining γ_a's satisfy

$$\dot\gamma_i=\tfrac12\sum_{j,k}\varepsilon_{ijk}[\gamma_j,\gamma_k]\,, \qquad (8)$$

where $i,j,k=1,2,3$ (or x,y,z) [i.e., $\gamma_x=(\gamma_\beta+\gamma_{\bar\beta})/2$, etc.]. The Nahm and Chazy equations are obtained from (8) by taking the γ_a's in su(2). It is convenient to work with the representation of SU(2) in $\mathbf{S}^3$ and express the γ_a's in terms of the left-invariant vector fields:

$$X_1=\cos\psi\,\partial_\theta+(\sin\psi/\sin\theta)\partial_\phi-\cot\theta\sin\psi\,\partial_\psi\,,$$

$$X_2=-\sin\psi\,\partial_\theta+(\cos\psi/\sin\theta)\partial_\phi-\cot\theta\cos\psi\,\partial_\psi\,,$$

$$X_3=\partial_\psi\,,$$

where θ, ϕ, and ψ are the Euler angles and $[X_i,X_j] = \sum_k \varepsilon_{ijk} X_k$. Setting $\gamma_i = \omega_i(t) X_i$ (no sum) in (8) one obtains the Nahm equations: $\partial_t \omega_i = \omega_j \omega_k$, $i \neq j \neq k$ and cyclic. Next, consider the adjoint action in su(2): $\gamma_i \to \gamma_i' = g\gamma_i g^{-1}$, $g \in \mathrm{SU}(2)$. It is well known that this automorphism induces a homomorphism $\varphi(g) \in \mathrm{SO}(3)$ and the γ_i transforms like a triad in so(3) as

$$\gamma_i \to \gamma_i' = \sum_j O_{ij} \gamma_j = \sum_j O_{ij} \omega_j(t) X_j .$$

$O \in \mathrm{SO}(3)$ is expressed in terms of the Euler angles in the usual manner. Substituting the γ_i''s in (8) and taking into account the extra contributions from the Lie bracket [due the action of the vector fields X_i on $O_{ij}(\theta,\phi,\psi)$] one obtains

$$\partial_t \omega_i = \omega_j \omega_k - \omega_i(\omega_j + \omega_k), \quad i \neq j \neq k \text{ and cyclic}, \tag{9}$$

where we used the following:

$$\sum_{i,j,k} \varepsilon_{ijk} O_{ip} O_{jq} O_{kr} = \varepsilon_{pqr} ,$$

$$\sum_i \varepsilon_{ijk} \varepsilon_{imn} = \tfrac{1}{2} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) ,$$

$$X_i(O_{jk}) = \sum_p \varepsilon_{ikp} O_{jp} .$$

Indeed, (9) arises as the self-dual Einstein's equation for Bianchi-IX cosological models.[12,13] By taking $y = -2 \times (\omega_1 + \omega_2 + \omega_3)$, (9) yields

$$\dddot{y} = 2y\ddot{y} - 3\dot{y}^2 , \tag{10}$$

which is the classical Chazy equation.[6] Given a solution of $y(t)$, one can recover the ω_i's in a straightforward way. Equation (10) can be solved by introducing a new variable s such that $y(t(s)) = 6\partial_t(\ln z_1)$, $t(s) = z_2/z_1$, where z_1 and z_2 are two independent solutions of the hypergeometric equation: $s(1-s)z'' + \{\frac{1}{2} - 7s/6\}z' - z/144 = 0$. These facts are verified by direct calculation.

The mapping $t = t(s)$ takes the upper-half s plane into the interior of a spherical triangle with angles 0, $\pi/2$, and $\pi/3$. For a detailed description of this mapping, see for example, Nehari.[14] The mapping $t(s)$ can be analytically continued by successive reflections of the upper-half s plane about the real line and by corresponding inversions of the fundamental triangle across its sides to complimentary triangles. However, these triangles do not cover the whole t plane but (in the limit of successive inversions) "tile" a region of the t plane bounded by a circle C. The inverse mapping $S(0,\pi/2,\pi/3;t)$ (which is the Schwarzian triangle function) is single valued (for these choices of parameters), meromorphic, and possesses a natural boundary as it cannot be analytically continued beyond C. Since $S(t)$ is invariant under a Möbius transformation, $t' = (at+b)/(ct+d)$, this natural boundary C is movable. $t(s)$ is one of a variety of such mappings related to Fuchsian differential equations which play an important role in the uniformization of the Riemann surfaces.[15]

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