# Phase Slips and the Instability of the Fukuyama-Lee-Rice Model of Charge-Density Waves

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(Received 23 March 1990)

It is argued that the threshold transition that occurs in the phase-deformation model of charge-density-wave systems in an applied electric field is destroyed in all dimensions by phase slips (amplitude fluctuations). The one-dimensional case is demonstrated using an exactly soluble model.

PACS numbers: 72.15.Nj, 64.60.Ht, 71.45.Lr

Studying the dynamics of sliding charge-density waves (CDW's) yields insight into the competition between randomness and interactions. CDW's are characterized by an order parameter with an amplitude and a phase. Some CDW materials display a nonlinear current-voltage characteristic with a threshold voltage that is small compared to typical electronic energies, implying that the impurity pinning strength is much smaller than the CDW stiffness.[1] Since long-wavelength distortions of the phase can cost arbitrarily little energy, whereas amplitude fluctuations must cost an energy of the same order as the gap energy, many investigators have studied the Fukuyama-Lee-Rice (FLR) model, which has only phase degrees of freedom.[2]

The transition between the pinned and moving states of the FLR phase distortion model when an external force $F$ is applied at zero temperature is a dynamic critical phenomenon.[3] The velocity $v$ in the FLR model has the same time-averaged value everywhere. The transition can be characterized using $v$ as an order parameter; $v$ is strictly zero when $F$ is below the threshold field $F_T$ and obeys $v \sim (F-F_T)^\zeta$ for $F \gtrsim F_T$, where $\zeta$ is a dimension-dependent critical exponent. A correlation length that diverges at threshold can be defined.

However, we show here that this transition is not physically relevant, since real CDW's at threshold cannot be described using phase fluctuations only. For any finite pinning strength, the threshold state of the FLR model has infinite local energy density and hence is unstable to the formation of regions with phase slip. The phase slips lead to a spatially nonuniform time-averaged velocity and destroy the critical behavior. This result implies that the depinning transition is always either discontinuous or rounded in the infinite-volume limit for a physically relevant randomly pinned model of CDW's. The rounding is very small in more than two dimensions, but in one and two dimensions this effect leads to total destruction of the critical behavior. Our arguments are general and can be applied to many driven systems in which interactions and randomness compete, such as flux motion in type-II superconductors.

We show the relevance of phase slip by examining a model where amplitude fluctuations are not allowed and showing that the local energy density is unbounded. The argument is presented heuristically; details will be presented elsewhere.[4] Although the scaling arguments employed here do not depend on the details of the system, for definiteness we consider a system described by the coupled equations of motion[5]

$$\dot{x}_i = k \sum_{\delta,\mathrm{nn}} (x_{i+\delta} - x_i) - V_i \sin(x_i - \beta_i) + F\,. \tag{1}$$

These equations describe a $d$-dimensional system of overdamped particles, each connected by springs obeying Hooke's law to its nearest neighbors, in the presence of a random pinning force described by $V_i$ and $\beta_i$. Every spring has a spring constant of $k$. The variables $x_i$ are the positions of the particles, where the motion is constrained to be along the direction of the force $F$, which is independent of both position and time. The absence of phase slips is enforced by the fact that the spring force $k(x_{i,i+\delta} - x_i)$ is linear in the particle separation for all $\delta x_{i,\delta} \equiv x_{i+\delta} - x_i$. Physically, one expects that the spring force has a maximum value, so that for large enough $\delta x_{i,\delta}$ the spring breaks and Eq. (1) is no longer valid. Thus, for the model to apply, every nearest-neighbor pair must have bounded separation $\delta x_{i,\delta} < S_{\max}$ for some finite $S_{\max}$, so that the local elastic energy density is less than $\frac{1}{2} k S_{\max}^2$. We show that this condition cannot be satisfied everywhere for the model described by Eq. (1).

The argument relies on the fact that the surface-to-volume ratio of a region of size $L$ vanishes as $L$ gets large.[6] Different regions might want to move at different velocities but are prevented from doing so by the spring forces. However, the regions communicate via springs that are only on the boundaries of the regions, while fluctuations in the impurity concentration and hence local threshold field are volume effects. Therefore, the force exerted by each spring on the boundaries of the regions become arbitrarily large as $L$ becomes large.

More specifically, consider a system of infinite size in the presence of a force $F$ just below its threshold for dc motion, $F_T(\infty)$. Inside this system, select a compact region with linear dimension $L$, which has threshold field $F_T(L)$.[7] If $F_T(L)$ is less than $F_T(\infty)$, then there must be a force $[F_T(\infty) - F_T(L)]L^d$ exerted on the region through its boundary by the surrounding regions to keep it from moving. This force can only come from the springs at the boundary. However, since the number of springs at the boundary scales only as $L^{d-1}$, at least one spring must be stretched by an amount that scales as $L$. The largest strain at the boundary (denoted here as $\delta x_{\max}$) is at least

$$\delta x_{\max} \sim L \Delta F_T(L) , \tag{2}$$

where $\Delta F_T(L) = F_T(\infty) - F_T(L)$. Equation (2) makes it obvious that a given (bounded) fluctuation in $F_T(L)$ leads to large strains when $L$ is large.

There are two types of fluctuations that must be considered; the first is exponentially rare in $L$ and leads to $\Delta F_T(L)$ of order $F_T(\infty)$. These rare fluctuations lead to a very small rounding of the threshold transition for all $d$, and are discussed in more detail below. The second is a typical $\sqrt{N}$-type fluctuation that leads to $\Delta F_T(L) \sim L^{-d/2}$. The strain induced by these typical fluctuations is at least $L\Delta F_T(L) \sim L^{1-d/2}$. This strain becomes larger than any finite $S_{\max}$ as $L \to \infty$ for $d<2$.

In one dimension this heuristic argument implies $\delta x_{\max} \sim L^{1/2}$. This result can be displayed analytically if one examines a simpler model first introduced by Mihaly, Crommie, and Gruner.[8] In this model the nonlinear pinning potential is replaced by a random static friction. The critical behavior (i.e., the $v$ vs $F$ relation) differs from that for Eq. (1), but the physics of the competition between randomness and interactions is retained. The equations of motion are

$$\begin{aligned} f_j &= k(x_{j+1} - 2x_j + x_{j-1}) + F - d_j , \\ \dot{x}_j &= f_j \text{ if } f_j > 0 , \\ \dot{x}_j &= f_j + 2d_j \text{ if } f_j < -2d_j , \\ \dot{x}_j &= 0 \text{ otherwise} . \end{aligned} \tag{3}$$

The $x_j$ are the positions of the particles, $F$ is the uniform force, and the $d_j$ describe the static friction, which is assumed to have a random component.

By summing over $j$, it is seen that the threshold field for this model is $F_T = \langle d_j \rangle$, where the angular brackets denote a spatial average. Thus, at $F=F_T$, $h_j \equiv (F - d_j)/k$ is a random variable with zero mean. If one defines new variables $\alpha_j$ and $\Delta_j$ which satisfy $\alpha_{j+1} - \alpha_j = h_j$ and $\Delta_j \equiv x_{j+1} - x_j + \alpha_{j+1}$, then just at threshold each $f_j = 0$ and $\Delta_j - \Delta_{j-1} = 0$ for all $j$, so that $\Delta_j$ must be independent of $j$. It is readily verified that $\alpha_j = \sum_{m=1}^{j-1} h_m + \alpha_0$, where $\alpha_0$ is an arbitrary constant that can be taken to be zero, so that

$$x_{j+1} - x_j = -\left( \sum_{m=1}^{j} h_m \right) . \tag{4}$$

Since the $h_i$ are random variables with zero mean, the *separations* obey a random walk. Thus, for a system of size $L$, the maximum particle separation obeys $\delta x_{\max} \sim L^{1/2}$. Thus, this exactly soluble case yields results consistent with the scaling arguments given above.

For $d=2$ it is straightforward to show that the spatial average $\langle (x_{i,j+1} - x_{i,j})^2 \rangle \propto \ln L$ for the random friction model at threshold.[9] For $d>2$, although a typical fluctuating region need not have a region of large strain at its boundary, it can be shown rigorously that rare fluctuations of $F_T(L)$ will still cause unbounded strains.[10] This result follows because $P(F_T(L))$, the probability of observing any value $F_T(L) > 0$, is nonzero; it depends roughly as[11]

$$P(F_T(L)) \sim \exp(-\{[F_T(\infty) - F_T(L)]L^d\}^2 / 2F_T(\infty)L^d) . \tag{5}$$

These rare regions with very small $F_T(L)$ are similar to those that lead to Griffiths singularities in the context of dilute ferromagnets[12] and spin glasses.[13] However, in contrast to the situation for random magnets, for the CDW system these rare fluctuations destroy the critical behavior.[14]

Estimates of the diverging strain can be made using a method similar to that of Randeria, Sethna, and Palmer[13] to discuss magnetic relaxations in spin glasses. A region of size $L$ in an applied force $F^*$ has strain greater than $S_{\max}$ if $F_T(L) < F^* - S_{\max}/L$. The number of regions $n_r$ with $F_T(L)$ less than this bound is

$$n_r \sim \int dL \int_0^{F_1} P(F_T(L)) , \tag{6}$$

where $F_1 = \max(0, F^* - S_{\max}/L)$. The integral must be evaluated keeping in mind the constraint $F_T(L) > 0$. For $d>2$ the integral is always dominated by the smallest values of $L$, near $L^* = S_{\max}/F^*$, leading to $n_r \sim \exp\{-[F_T(\infty)(S_{\max}/F^*)^d/2]\}$. Thus, $n_r$ is nonzero for any $F^* > 0$.

So far it has been shown that the model described by Eq. (1) has unbounded particle separations. Now it is argued that if the spring force is bounded then the time-averaged velocity must be nonuniform. For simplicity, consider a region $R$ of size $L$ with no impurities, so that the random potential term vanishes inside the region. If one requires the spring force to be less than some bound $kS_{\max}$, then averaging the analog of Eq. (1) over the region $R$ shows that $v_R$, the spatially averaged velocity in $R$, must satisfy

$$v_R L^d \geq F L^d - S_{\max} L^{d-1} . \tag{7}$$

For any given $F>0$, by finding a large enough region, one can bound $v_R$ to be strictly greater than zero. Similarly, any region with $F_T(L) < F_T(\infty)$ will move if it is large enough.[15] A more heuristic argument is to consider what happens when the strain at the boundary of the region exceeds $S_{\max}$. For simplicity assume that if $\delta x_{\max} > S_{\max}$ the spring breaks, so that it no longer exerts any force.[16] Now the force needed to keep the region stationary must be applied by fewer springs, so that repeating the above argument indicates that another spring must break. Iterating this process shows that the region must break free. This line of reasoning leads one to conclude that the depinning of the region is discontinuous, so that the velocity of the region jumps by $(S_{\max}/L)^{\zeta}$, where $\zeta$ is the exponent describing the velocity characteristic in the model with no phase slip ($S_{\max} \to \infty$) and $L$ is the linear dimension of the region.[17] The first-order nature of the depinning of a region is also supported by mean-field and single degree-of-freedom calculations on related models.[18]

We now show that these results imply that phase slips destroy the FLR critical behavior in all dimensions. For $d=1$ and 2, as the putative threshold is approached, the system breaks up into disconnected pieces and the FLR model fails completely. For $d \gtrsim 3$ a connected region of the CDW could depin at a well-defined threshold field, but when the correlation length of the FLR model reaches the typical separation of moving regions the FLR model should cease to be a good description. One scenario is that phase slips drive the depinning discontinuous. This possibility is supported by the fact that the connected region must break free from regions that have anomalously strong pinning and are hence stationary for $F > F_T(\infty)$, so that a nonzero density of springs must break at "threshold." A second possibility (weakly supported by the experimental evidence cited below) is that the phase slips act as effective noise sources and round the transition. In either case the critical behavior is destroyed.

The obvious experimental consequence of this work is that FLR critical behavior at the CDW threshold transition is never seen, even in perfectly homogeneous samples. However, the effects discussed in this paper on the CDW velocity in three-dimensional systems are very small.[19] On the other hand, this work may shed light on the discrepancy between the experimental observation of broadband noise in CDW's above threshold[20] and the lack of such noise in the FLR model.[21] Arguments similar to those presented above suggest that the FLR model is an incomplete description of the CDW for fields above the FLR threshold. Stationary and moving regions coexist, so that phase slips must occur. It has been suggested already that phase slip and broadband noise are intimately connected,[22] and this work indicates that the experimental observations are not caused by inhomogeneous samples, but result from intrinsic properties of the system. This conclusion is in accord with the reproducibility of the broadband noise observations and the fact that strong ac fields that cause mode-locking suppress the broadband noise.[23] However, detailed experimental predictions require better understanding of the phase-slip dynamics.

This work may have interesting relations to other models of nonlinear systems with many degrees of freedom. The results here are consistent with simulations of two-dimensional magnetic-flux lattices with impurities that indicate that grain boundaries occur even for weak pinning, and that these regions dominate the resistance caused by flux creep at low temperatures.[24] This work may be related to that of Carlson and Langer on a model of earthquakes;[25] both models display a buildup of large strains and catastrophic events. Systems exhibiting self-organized criticality[26] also have features similar to those found in CDW models. However, these relationships remain to be elucidated.

In conclusion, this paper has shown that a CDW in an electric field must be described using both phase and amplitude fluctuations. When this is done, the critical behavior at threshold present in the phase-only model is destroyed.

This work grew out of many invaluable conversations with M. Inui, P. B. Littlewood, and A. J. Millis. The author acknowledges discussions with J. Carlson, E. Grannan, and G. Swindle and thanks S. Strogatz for pointing out Refs. 9 and 10.

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