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Calculation of Giant Fractional Shapiro Steps in Josephson-Junction Arrays

K. H. Lee, D. Stroud, and J. S. Chung

Department of Physics, The Ohio State University, Columbus, Ohio 43210

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We calculate the response of an $N \times N$ array of resistively shunted Josephson junctions to an imposed current $I = I_{dc} + I_{ac} \sin(2\pi \nu t)$. In a transverse dc magnetic field of $p/q \equiv f$ flux quanta per plaquette of area, we find fractional giant Shapiro steps in the time-averaged voltage $\langle V \rangle$ at values $\langle V \rangle = nNh\nu/2eq$, $n = 1, 2, 3, \ldots$, in agreement with the measurements of Benz *et al.* At $f = \frac{1}{5}$, $\frac{2}{5}$, and $\frac{1}{3}$, we find additional fractional steps at $\langle V \rangle = Nh\nu/4e$. A generalization of the model of Benz *et al.* accounts for both the fractional giant steps at p/q and the anomalous half-integer steps.

When a resistively shunted Josephson junction is subjected to a combined dc and ac current $I(t) = I_{dc} + I_{ac} \times \sin(2\pi \nu t)$, the time-averaged voltage $\langle V \rangle$ across the junction exhibits plateaus, known as Shapiro steps,[1] at multiples of $h\nu/2e$. The height of Shapiro steps provides an extremely accurate means of measuring the fundamental ratio h/e.

In this Letter, we calculate the response of an $N \times N$ square Josephson-junction network[2] to a combined dc and ac current. In a dc transverse magnetic field $H = (p/q)\Phi_0/a^2$, where $\Phi_0 = hc/2e$ is the flux quantum, a is the lattice constant of the Josephson-junction network, and $p/q \equiv f$ is the ratio of two mutually indivisible integers, the time-averaged voltage drop $\langle V \rangle$ across the array is found to exhibit steps at intervals of $(N/q)h\nu/2e$. Our I-V characteristics agree quite well with those measured by Benz *et al.*[3] At $f = \frac{1}{5}$, $\frac{2}{5}$, and $\frac{1}{3}$, we find additional fractional steps at $\langle V \rangle = (N/2)h\nu/2e$. While these extra fractions are not seen in Ref. 3, they might be found at lower temperatures, less disorder, or a different ac amplitude or frequency than those studied experimentally. We generalize the phenomenological model of Benz *et al.* to account for both the steps found by them and our anomalous half-integer steps.

Our calculation proceeds by directly solving the equations for a network of resistively coupled Josephson junctions in the limit of zero shunt capacitance and negligible array self-inductance:[4]

$$I_{ij} = V_{ij}/R_{ij} + I_{c;ij} \sin(\phi_i - \phi_j - A_{ij}), \qquad (1)$$

$$V_{ij} = \frac{\hbar}{2e} \frac{d}{dt}(\phi_i - \phi_j), \qquad (2)$$

$$\sum_j I_{ij} = I_{i;ext}. \qquad (3)$$

Equation (1) describes the current from grain i to grain j as the sum of a normal contribution V_{ij}/R_{ij} and a Josephson current. Equation (2) is the Josephson relation connecting the voltage difference V_{ij} between grains i and j and the phase difference $\phi_i - \phi_j$ between the phases of the order parameters. Finally, Eq. (3) is

Kirchhoff's law, expressing current conservation at grain i. The given form of the Josephson current is appropriate in a transverse magnetic field $\mathbf{B} = \mathbf{V} \times \mathbf{A}$. The factor

$$A_{ij} = (2\pi/\Phi_0) \int_{\mathbf{x}_i}^{\mathbf{x}_j} \mathbf{A} \cdot d\mathbf{x},$$

where $\mathbf{x}_i$ is taken as the center of grain i. We assume a square array of $N \times N$ junctions [or $(N+1) \times (N+1)$ grains]. A current $I = I_{dc} + I_{ac} \sin(2\pi \nu t)$ is fed into each grain in the top row and extracted from each grain on the bottom row, with free boundary conditions on the two transverse boundaries. Combining Eqs. (1)-(3) yields coupled first-order nonlinear differential equations for the phases, which can be solved iteratively. We have carried out this iteration using time steps of 0.01τ to 0.02τ, where $\tau = \hbar/2eRI_c$, R being the shunt resistance and I_c the critical current of each junction.

Figure 1 shows representative voltage traces for 12×12 and 10×10 arrays at several values of the flux per plaquette f, measured in units of Φ_0, using $I_{ac} = I_c$, and $\omega/\omega_0 = 0.1$, where $\omega = 2\pi \nu$ and $\omega_0 = 2\pi/\tau$. This value of I_{ac} was chosen to correspond to the experimental conditions of Ref. 3; but we found numerically that the same Shapiro steps were also present for other, greater or lesser, values of I_{ac}, as well as in disordered samples with random I_c's. The time-averaged voltage $\langle V \rangle$ shown is the difference between the mean voltages along the top and bottom rows, typically averaged over the time interval of 800τ. The I-V characteristic is calculated at intervals of $0.015I_c$, the current being ramped up after each I-V point is evaluated. For initial conditions, we have used parallel phases for the first point calculated, with subsequent points obtained using as an initial state the final phase configuration of the previous point. Other initial conditions were tried, and generally produce only slight differences in the I-V characteristics, except for somewhat broadening the riser to the first step at $f = 0$.

For all fields shown, there are characteristic plateaus in $\langle V \rangle$, generally with spacings of $Nh\nu/2eq$. The $q = 1$ plateaus are usually much wider than those at higher q. Also, the principal ($n = 1$) giant step is far
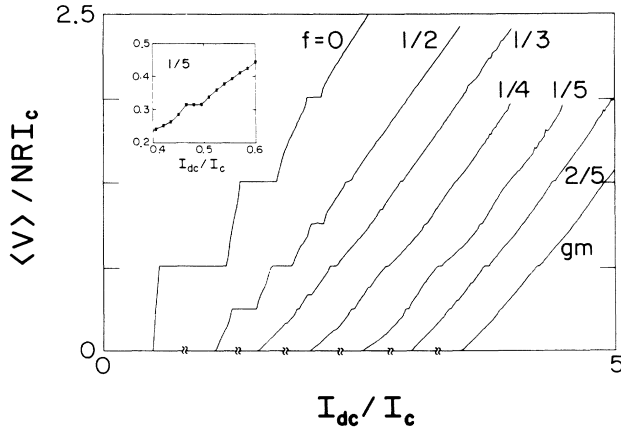
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FIG. 1. Time-averaged voltages $\langle V(t) \rangle$ vs dc current I_{dc} in an $N \times N$ array of Josephson junctions at several values of the transverse magnetic field. f is the flux per plaquette in units of a flux quantum $\Phi_0 = hc/2e$. $N = 12$ for all curves shown except $f = \frac{1}{5}$ and $\frac{2}{5}$, for which $N = 10$. The notation "gm" refers to a flux per plaquette $f = 1 - (\sqrt{5} - 1)/2$. In all cases, there is an ac current $I_{ac} \sin(2\pi v t)$ with $I_{ac} = I_c$ and $v = 0.1(2eRI_c/\hbar)$, where I_c is the critical current of a single junction and R is the shunt resistance. All except the $f = 0$ curve are horizontally displaced; except for $f = 0$, the critical current for the onset of nonzero voltage is about 0.05. Inset: Expansion of half-integer step at $f = \frac{1}{5}$.

FIG. 2. Time-averaged voltage $\langle V \rangle$ for a disordered 12×12 array of Josephson junctions subjected to an applied current as in Fig. 1. c is the fraction of junctions with critical current I_c; the remainder have $0.5 I_c$. T is the temperature in units of $\hbar I_c/k_B e$.

broader than the higher-n plateaus. For $f = 1 - g$, where $g = (\sqrt{5} - 1)/2$ is the golden mean, the differential resistance dV/dI is nonmonotonic, and seems to show a precursor of an integer giant step at $Nhv/2e$. When $f = \frac{1}{5}$, $\frac{2}{5}$, and $\frac{1}{3}$, we find "anomalous" half-integer steps at $(N/2)hv/2e$. These steps are not artifacts of the calculation, as is shown in the inset of Fig. 1 where we show an enlarged portion of the I-V characteristic for $f = \frac{1}{5}$.

The Shapiro steps are rounded by both disorder and finite temperature. To introduce disorder in the critical currents (but not the shunt resistances), we allow a fraction $1 - c$ of the junctions, chosen at random, to have critical current $0.5 I_c$, while the remainder have I_c. At both $f = 0$ and $f = \frac{1}{2}$, we find that the width of the steps is reduced, for fixed values of v and I_{ac}, while the edges of the steps are slightly rounded, and at $f = 0$ the riser to the first giant step is somewhat broadened. We include temperature by adding to each junction a parallel Langevin-noise current source $I_L(t)$ (Ref. 5) with a Gaussian distribution whose ensemble average satisfies $\langle I_L(t) \rangle_e = 0$, $\langle I_L(t)I_L(t') \rangle_e = (2k_B T/R)\delta(t - t')$, and noise currents in different junctions assumed uncorrelated.[4] As shown in Fig. 2, such temperature noise at $f = 0$ significantly rounds the steps. We have found similar behavior at $f = \frac{1}{2}$ for both disorder and finite temperature.

In a single Josephson junction, Shapiro steps result from mode locking. On the steps, the voltage is a periodic function of time, corresponding to a synchronism be-
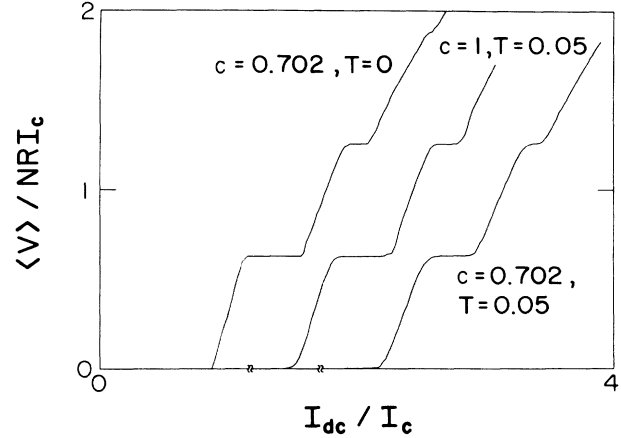
tween the applied ac field and the natural frequency of the voltage across the Josephson junction. Between the steps, however, these two modes do not lock and the voltage is aperiodic. Both the integer and the fractional giant steps of Fig. 1 exhibit similar behavior, as is illustrated in Fig. 3 for $f = \frac{1}{2}$. We find that other values of f behave similarly, even in the presence of disorder. In particular, the half-integer step at $f = \frac{1}{5}$ is characterized by a periodic voltage signal.

Next, we generalize the model of Benz et al.[3] to allow for additional fractions beyond those permitted in their
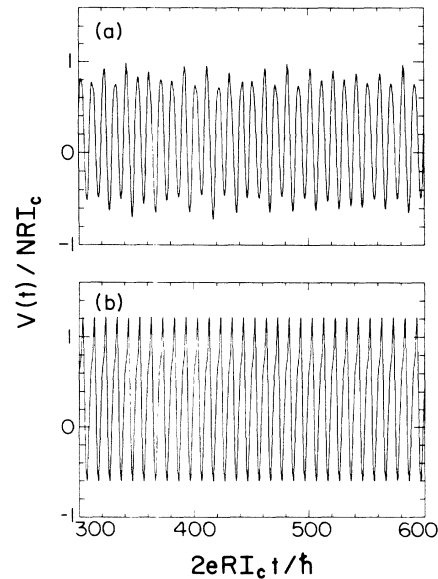

FIG. 3. Voltage traces $V(t)$ for a 12×12 array at $f = \frac{1}{2}$, as in Fig. 1, with (a) $I_{dc} = 0.20$ (no step); (b) $I_{dc} = 0.46$ (step at $Nhv/4e$).

model. The *static* properties of Josephson-junction arrays, in the absence of an applied current, are often described[6] in terms of the so-called frustrated xy model. This model is equivalent, via a Villain transformation, to a classical Coulomb gas of "vortices" of charge f and $1-f$. The charges are constrained to sit in the centers of the plaquettes formed by the grains, have interactions which are logarithmic at sufficiently large separations, and are of such numbers as to insure charge neutrality.

Following Teitel and Jayaprakash,[6] we assume that, in a transverse magnetic field of p/q flux quanta per plaquette, the ground state of a square lattice has a unit cell of $q \times q$ plaquettes. The plaquettes contain vortices of charge p/q and $1-p/q$, arranged so as to insure charge neutrality and minimum energy. When a dc current is applied to this lattice from top to bottom, it drives the positive charges to the left and the negative ones to the right. If the current is large enough,[3] one possibility is that the lattice will be depinned *as a whole* from the underlying "egg-carton" pinning potential, producing a net flow of positive charge to the left, and hence, a voltage drop from top to bottom. This flow can be visualized as a series of processes in which a positive and an adjacent negative vortex change places. With each interchange, there is a phase slip of magnitude 2π (since the total charge moving across the vertical bond separating positive and negative charge is unity).

To illustrate the emergence of fractional steps, we consider $f = 1/q$. The smallest rigid motion of the vortex lattice which will return it to an equivalent energy minimum[3] is one array lattice constant, a distance a. q is the number of charge interchanges, per $q \times q$ unit cell, required to accomplish this motion. If these occur in time t_0, then there will be one phase slip per row of length q in time t_0 [see Fig. 4(a), where this process is illustrated for the case $q = 5$]. The voltage drop along such a row will be $2\pi\hbar/2et_0$, or $(N/q)2\pi\hbar/2et_0$ across an $N \times N$ array. To produce a Shapiro step, the vortex lattice must move a multiple of the distance a per cycle of the ac field, so that the lattice motion can lock onto the egg-carton potential. This requirement is equivalent to $1/t_0 = nv$, or $V = nNhv/2eq$, where n is an integer. This argument accounts for the fractional steps at multiples of $Nhv/2eq$, but similar arguments can explain at least some of the ratios p/q where $p > 1$.

To account for the half-integer giant steps at $f = \frac{1}{5}$ and $\frac{1}{3}$, we propose that at such fields, the vortex lattice is not necessarily depinned *as a unit*, but instead, at certain currents, a sublattice consisting of every *second* row of vortices moves as a unit to the next equivalent energy minimum in a single cycle of the ac current.[7] During each cycle, half the positive charges move a distance qa to the left, while the other half do not move. (During the next cycle, presumably the other sublattice moves in a similar fashion.) In a $q \times q$ unit cell, there will be $q/2$ phase slips per row of length q in time $t_0 = 1/v$, and,
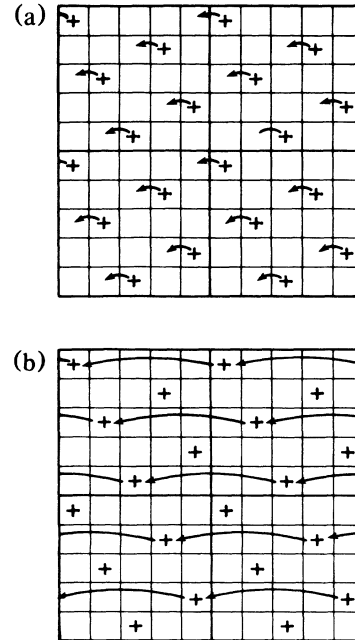

FIG. 4. Ground-state vortex configuration for $f = \frac{1}{5}$, after Teitel and Jayaprakash. The + symbols denote positive vortices of charge $\frac{4}{5}$; other plaquettes contain vortices of charge $-\frac{1}{5}$. Arrows in (a) denote the pattern of positive-vortex motion required to move entire vortex lattice a distance a to the left, as postulated by the model of Ref. 3. This motion produces a voltage step at $\frac{1}{5}Nhv/2e$. Arrows in lower picture show motion required to move one vortex sublattice a distance $5a$ to the left, as proposed in the present paper. This motion produces a step at $\frac{1}{2}Nhv/2e$.

therefore, a half-integer giant step at $(N/2)hv/2e$ across an $N \times N$ lattice. This process is illustrated in Fig. 4(b).

Intuitively, this kind of phase slip seems energetically favorable. The motion involves exciting a high-wavenumber distortion of the vortex lattice in the presence of the periodic pinning potential. Such a mode might cost less energy to excite than a motion of the lattice as a whole. Moreover, such distortion modes may also occur at other values of $f = p/q$. If the unit cell is $q \times q$, such a sublattice motion of this kind would also lead to a half-integer giant step for such values of q. Accordingly, we propose that, in addition to those found by Benz *et al.*, there are also steps at $(N/2)hv/2e$ for other odd values of q.

The picture proposed here is obviously only the initial stage of a detailed model for fractional giant Shapiro steps, which may have implications for the use of such arrays as coherent emitters or absorbers of radiation.[8,9] There may be many other complex instabilities of a vortex lattice besides those discussed here, each giving rise to its own signature of Shapiro steps. For example, if every *third* row moved by qa, the voltage drop would be $(N/3)hv/2e$, and so on. There are slight hints of other

such "anomalous" steps even in our rather small-lattice calculations. Even the "nonanomalous" fractional steps at $nNhv/2eq$ could be accounted for not only by rigid-lattice motions but also by modes involving various sub-lattices of the vortex lattice. For example, some of them could be produced by motion of every second row at a field with even q. Such a motion would generate only nonanomalous fractions, indistinguishable from those predicted by the model of rigid-lattice motion.

Note added.—Similar calculations have recently been carried out by J. U. Free, S. P. Benz, M. Rzchowski, M. Tinkham, and C. J. Lobb (to be published).

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