# Instabilities of One-Dimensional Cellular Patterns

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(Received 11 October 1989)

Symmetry arguments are used to classify the various modes of instabilities of one-dimensional periodic patterns. An important feature of this theory is to point out the coupling between these modes and the phase of the cellular structure. A number of results presented allow us to interpret recent observations in hydrodynamical-flow and directional-solidification and -fingering experiments.

PACS numbers: 47.10.+g, 05.45.+b, 47.20.Ky

Recent experiments, including Couette-Taylor flow,[1] Rayleigh-Bénard flow,[2] directional-solidification,[3-6] and directional-fingering experiments,[7] report a number of very interesting observations on secondary instabilities of one-dimensional cellular patterns.

The aim of this Letter is to describe, on the basis of symmetry arguments, generic instabilities of such periodic patterns. Such an instability can be either stationary or oscillatory. Its spatial period can be the same, double, or "irrationally" related to the period of the basic structure. In the first two cases only the parity of the critical eigenmode matters. We are led to define ten different generic cases. Remarkably enough this simple classification leads to a very rich variety of patterns, including breathing, vascillation, and translation of the basic cellular pattern. Spatial fluctuations allow the existence of localized solutions, defects, and phase instabilities which have a specific character, because of the coupling with the phase of the underlying periodic pattern.

Our analysis starts with a partial differential equation which describes a physical system extended in only one spatial dimension,

$$\partial_t U = f_\mu(U,\nabla) , \tag{1}$$

where $\mu$ represents a typical control parameter, dropped in the following. This system is assumed to be invariant under the following transformations: $T_h$, $x \to x+h$ (space translation); $S$, $x \to -x$ (parity symmetry); and $\mathcal{T}_\theta$, $t \to t+\theta$ (time translation). Our fundamental hypothesis is the existence of a stable, steady ($\mathcal{T}_\theta U_0 = U_0$), $x$-periodic solution of Eq. (1) ($T_a U_0 = U_0$, where $a$ is the spatial period) which is invariant under the parity symmetry $S$ ($SU_0 = U_0$). Although $U_0$ generally depends on the transverse coordinates, we have dropped these variables for the sake of clarity. We first remark that the basic pattern only breaks $x$ translation. It is thus invariant under the subgroup of transformations $T_{na}$ (for any integer $n$), $S$, and $\mathcal{T}_\theta$ (for any real $\theta$). Instabilities of this pattern are likely to break some of these symmetries. Before addressing the stability problem, a few remarks are in order. Thanks to the $x$-translation invariance, for any constant $\phi$, $U_0(x+\phi)$ is also a solution of Eq. (1); then $\xi_0(x) = \partial U_0/\partial x$ appears as a marginal mode of the linear problem associated with the stability of $U_0$. This marginal mode describes phase perturbations of the basic pattern. This degree of freedom is in general coupled with the possible instabilities of the cellular pattern. Let us look for a perturbation of $U_0$ under the form $U(x,t) = U_0(x+\phi) + u(x+\phi,t)$, where $u$ is chosen to be orthogonal to $\xi_0$, i.e., $(u,\xi_0) = 0$, where $(f,g)$ denotes the scalar product in phase space. After linearization Eq. (1) reduces to

$$\partial_t \phi = (\mathcal{L}u, \chi_0) , \tag{2}$$

$$\partial_t u = \mathcal{L}(x)u - (\mathcal{L}u,\chi_0)\xi_0(x) \equiv L(x)u , \tag{3}$$

where $\chi_0(x) \equiv \xi_0(x)/||\xi_0||^2$ and $\mathcal{L}(x) = \partial f/\partial U|_{U_0}$. $\mathcal{L}$ and $L$ are linear operators with periodic coefficients which commute with $T_a$, $\mathcal{T}_\theta$, and $S$. The stability of $U_0$ reduces to the study of the eigenvalue problem $LV = \sigma V$. Let us assume that for the parameter value $\mu = 0$, an instability occurs, i.e., there is a critical eigenvalue $\sigma_0 = 0$ or $\sigma_0 = \pm i\omega$. Since the eigenvalue problem is a linear differential equation with periodic coefficients, a general bounded solution can be found under a Floquet form (in solid-state physics, it is known as the Bloch wave function): $V_0(x) = \exp(ikx)\hat{V}_0(x)$, where $\hat{V}_0(x+a) = \hat{V}_0(x)$, and $k$ is a real constant. Generically, a solution $V$ of Eq. (3) has either the same period $a$ (i.e., $k=0$) as the basic cellular pattern, the double period $2a$ (i.e., $k=\pi/a$), or a period irrationally[8] related to $a$. Depending on the dimension of the kernel of $L_0 - \sigma_0 I$ several amplitudes can be necessary to describe the weakly nonlinear supercritical regime. Let us denote by $V_0^j$ the critical eigenvectors of $L$ for $\mu = 0$. Near $\mu = 0$, at the leading linear order, a solution of Eq. (1) is looked for under the form

$$U(x,t) = U_0(x+\phi) + u(x+\phi,X,t,T) + \cdots ,$$

where

$$u(x,X,t,T) = \sum_j A_j \exp(\sigma_0^j t) V_0^j(x) , \tag{4}$$

where $A_j$ and $\phi$ are functions of *slow variables* $(X,T)$, and where the ellipsis expresses the slave variables in terms of $A_j$ and $\phi$. $U(x,t)$ represents a solution of Eq. (1) if the amplitudes $A_j$ and $\phi$ satisfy solvability conditions

$$\partial_T A_i = \mathcal{A}_i(\{A_j\},\phi)\,,\quad \partial_T\phi = \Phi(\{A_j\},\phi)\,. \tag{5}$$

We show below the specific form of Eqs. (5) for the ten different generic cases which describe all the possible instabilities occurring in a one-parameter family of dynamical systems.

*(A) Stationary instabilities.*— (1) At the same spatial period ($k=0$), the perturbation $u$ reads

$$u(x,X,t,T) = A(X,T)\hat{V}_0(x)\,. \tag{6}$$

First we look at the *symmetrical case*, $S\hat{V}_0 = \hat{V}_0$. Applying to Eq. (6) the subgroup of transformations which leaves $U_0$ invariant, one discovers the corresponding representations of the amplitude $A$ (real order parameter) and the phase $\phi$. The principal part of the equivariant amplitude equation after an appropriate scaling now reads

$$A_T = \mu - A^2 + \xi_1 A_{XX} + \xi_2\phi_X\,, \tag{7a}$$

$$\phi_T = A_X + \phi_{XX}\,, \tag{7b}$$

where the subscripts $T$ and $X$, respectively, represent the derivative with respect to $T$ and $X$, and $\xi_i$ with $i=1,2$ are real constants. This case corresponds to a saddle-node bifurcation where the basic pattern disappears for negative values of $\mu$.

In the *antisymmetrical case*, $S\hat{V}_0 = -\hat{V}_0$, the system has to respect the invariance $x\to -x$, $A\to -A$, $\phi\to -\phi$; the generic principal part now takes the form

$$A_T = \mu A \pm A^3 + \xi_1 A_{XX} + \xi_2 A_X A + \xi_3\phi_X A + \xi_4\phi_{XX}\,, \tag{8a}$$

$$\phi_T = A + \phi_{XX} + \xi_5 A_{XX}\,. \tag{8b}$$

This is a very interesting and very common case since the breaking of the parity symmetry, whose $A$ is the order parameter, induces, thanks to the first term in Eq. (8b) [due to the antisymmetry of the Goldstone mode $\xi_0(x)$], a translation of the cellular pattern itself. This parity-breaking transition has been observed in particular in the case of transversally driven interfaces,[3-7] where it appears under the form of tilted cells which translate at a constant velocity [see Figs. 1(a)-1(c)]. These phenomena also appear in binary convection,[9,10] in the unfolding of codimension-two bifurcations,[11,12] and in model equations.[13] These equations also describe source and sink defects, which correspond to points where the basic pattern translates in opposite directions.[14] In the case of a subcritical symmetry-breaking transition,[15] droplets of asymmetric cells propagate inside the basic pattern.

(2) At double the spatial period ($k=\pi/a$), the perturbation reads

$$u(x,X,t,T) = A(X,T)V_0(x)\,, \tag{9}$$

where $V_0(x) = \exp(i\pi x/a)\hat{V}_0(x)$ is such that $V_0(x+a) = -V_0(x)$ and is *real*.

In the *symmetrical case*, $SV_0 = V_0$, we have

$$A_T = \mu A \pm A^3 + \xi_1 A_{XX} + \xi_2\phi_X A\,, \tag{10a}$$

$$\phi_T = \partial_X(A^2) + \phi_{XX}\,. \tag{10b}$$

This case is very similar to the period-doubling bifurcation for time-periodic solutions of ordinary differential equations.

The other case is the *antisymmetrical case*, $SV_0 = -V_0$. Although this case is physically distinct from the previous one [see Figs. 2(a) and 2(b)], the amplitude equation has the same form as Eqs. (10). Spatial period-doubling bifurcation has been in particular observed in directional-solidification experiments.[6]

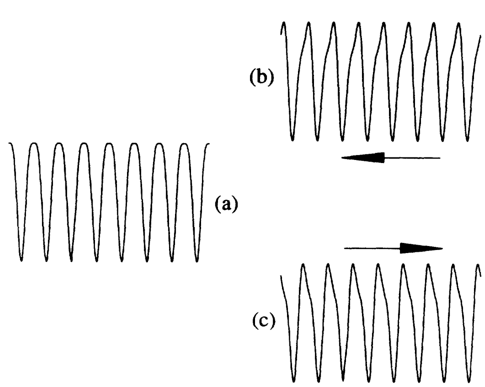


FIG. 1. Sketch of the parity-breaking instability. (a) The basic pattern has been chosen in order to mimic a solidification interface (see, for example, Ref. 3). (b) and (c) represent, respectively, the broken-symmetry states which respectively propagate to the left and to the right.

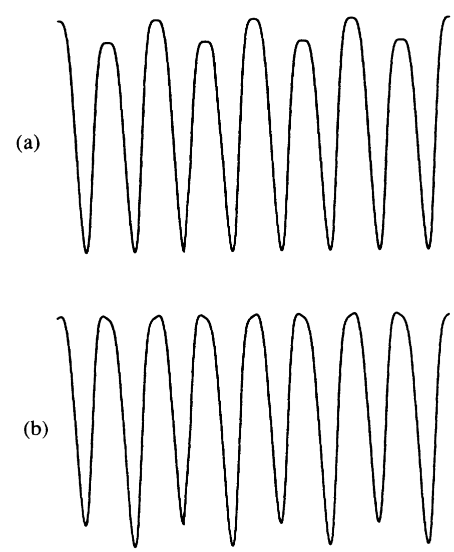


FIG. 2. Sketch of the stationary period-doubling instability. (a) Symmetrical case. (b) Antisymmetrical case.

(3) At a spatial period that is irrationally[8] related to $a$, the perturbation $u$ reads

$$u(x,X,t,T)=A(X,T)\exp(ikx)\hat{V}_0(x)+\text{c.c.}, \quad (11)$$

where c.c. means complex conjugate. The amplitude equation reads

$$A_T=\mu A\pm|A|^2A+\xi_1A_{XX}+\xi_2\phi_XA, \quad (12a)$$

$$\phi_T=\partial_X(|A|^2)+\phi_{XX}, \quad (12b)$$

where $\xi_i$ with $i=1,2$ are *real* constants. No experimental evidence of this case has been reported so far.

*(B) Oscillatory instabilities.*— (1) At the same spatial period ($k=0$), the perturbation reads

$$u(x,X,t,T)=A(X,T)\exp(i\omega t)\hat{V}_0(x)+\text{c.c.} \quad (13)$$

For the *symmetrical case,* $S\hat{V}_0=\hat{V}_0$, since $\hat{V}_0$ and $U_0$ have the same period and the same symmetry, the instability leads to a "breathing-like" oscillation [see Fig. 3(a)]. The amplitude equation reads

$$A_T=\mu A-(\pm 1+i\alpha)|A|^2A+\xi_1A_{XX}+\xi_2\phi_XA, \quad (14a)$$

$$\phi_T=\partial_X(|A|^2)+i\beta(A_X\bar{A}-A\bar{A}_X)+\phi_{XX}, \quad (14b)$$

where $\alpha$ and $\beta$ are real, $\xi_i$ with $i=1,2$ are complex constants, and $\mu$ is complex.

For the *antisymmetrical case,* $S\hat{V}_0=-\hat{V}_0$, the amplitude equation is the same as for the symmetrical case. Physically the instability leads to a different spatiotemporal pattern, which can be seen as a "vascillation" of the basic pattern [see Fig. 3(b)]. In supercritical cases a phase instability can occur. This instability could be at the origin of the spatiotemporal intermittency.[7,16-18]

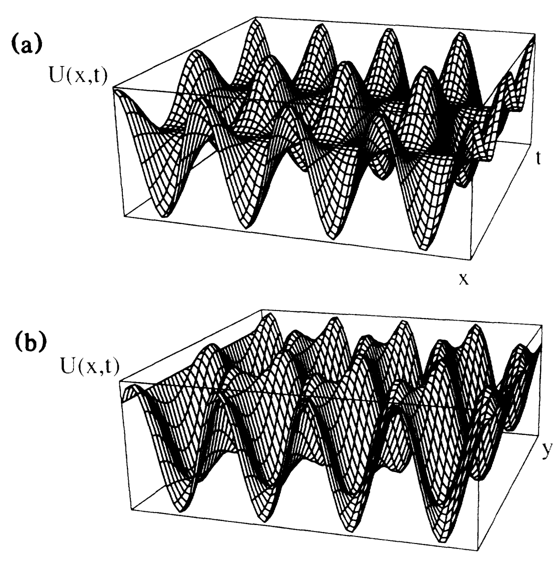


FIG. 3. Sketch of the oscillatory instability at the same spatial period. (a) Breathing mode. (b) Vascillation mode.

Both instabilities are observed in the Taylor-Couette problem.[1] The symmetrical case gives the so-called "twisted vortices," while the antisymmetrical case gives wavy vortices, both bifurcating from Taylor vortices [denoted here $U_0(x)$]. The symmetrical case was also observed in Rayleigh-Bènard experiments where it has been coined as vascillation.[2]

(2) At double the spatial period ($k=\pi/a$) the perturbation reads

$$u(x,X,t,T)=A(X,T)\exp(i\omega t)V_0(x)+\text{c.c.}, \quad (15)$$

where $V_0(x)=\exp(i\pi/a)\hat{V}_0(x)$ is such that $V_0(x+a)=-V_0(x)$.

The amplitude equation is the same as for the symmetrical or antisymmetrical case and it does not depend on the symmetry of $V_0$. This mode of oscillation mixes both breathing and vascillation. The two cases lead to physically different situations. They are observed, for instance, in the Taylor-Couette problem (WIB and WOB of Ref. 1, see also Ref. 19).

(3) At a spatial period that is irrationally[8] related to $a$, the perturbation reads

$$u(x,X,t,T)=A(X,T)\exp(i\omega t+ikx)\hat{V}_0(x)+B(X,T)\exp(i\omega t-ikx)S\hat{V}_0(x)+\text{c.c.} \quad (16)$$

The amplitude equations read

$$A_T=cA_X+\mu A-[(\pm 1+i\alpha)|A|^2+(\beta+i\gamma)|B|^2]A+\xi_1A_{XX}+\xi_2\phi_XA, \quad (17a)$$

$$B_T=-cB_X+\mu B-[(\pm 1+i\alpha)|B|^2+(\beta+i\gamma)|A|^2]B+\xi_1B_{XX}+\xi_2\phi_XB, \quad (17b)$$

$$\phi_T=(|A|^2-|B|^2)+\phi_{XX}+i\eta(A_X\bar{A}-A\bar{A}_X)+i\chi(B_X\bar{B}-B\bar{B}_X)+\delta\partial_X(|A|^2+|B|^2), \quad (17c)$$

where $\xi_j$ and $\mu$ are complex, and $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$, and $\chi$ are real. In the case $A=B$ (i.e., when $|\gamma|<1$) one observes a spatially quasiperiodic standing wave, while in the case of $A\neq 0$ (respectively $A=0$) and $B=0$ (respectively $B\neq 0$) (i.e., when $|\gamma|>1$), the resulting state is spatially and temporally quasiperiodic. The spatially quasiperiodic perturbation propagates with a velocity $\omega/k$ (respectively $-\omega/k$), while, thanks to the first term in Eq. (17c), the basic cellular pattern translates itself with a velocity $|A|^2$ (respectively $-|B|^2$). This kind of behavior could be related to some experimental observation in shear-flow experiments in circular geometries.[20]

This study is obviously related to the more classical frame of bifurcations under discrete groups of symmetries,[21] and in particular with bifurcations from group orbits.[22] The aim of this Letter was to give a short description of the ten generic instabilities [see Eqs. (6), (9), (11), (13), (15), and (16)] of one-dimensional periodic patterns, taking account of large-scale spatial modulations. Experimental results which find their interpretation in this framework are hydrodynamical instabilities (Couette-Taylor and Rayleigh-Bènard flows) and directional solidification and fingering. Our preliminary results lead to many interesting questions, including a possible relation between the Eckhaus instability of the cellular patterns triggered by one of the instabilities and the spatiotemporal intermittent destruction of spatial order. Let us finally mention that this kind of analysis can be generalized to higher-dimensional cellular patterns.

This work has been motivated by recent experiments by P. Bergé, J. J. Bechhoefer, Y. Couder, M. Dubois, A. Libchaber, S. Michalland, A. Simon, and M. Rabaud, who are thanked for interesting discussions. One of us (P.C.) wants to thank the Aspen Center for Physics where part of this work was completed. P. Hohenberg is thanked for criticism and comments on an earlier version of this work. We have also benefited from discussions with S. Fauve, L. Gil, J. Lega, and Y. Pomeau. This work has been supported by the European Economic Community under Contracts No. ST2J-0316-C and No. SC1-0035-C, and the DRET [Direction des Recherches Etudes et Techniques (Contract No. 312/88)]. Laboratoire de Physique Théorique is Unité Associé au CNRS No. 767 and Laboratoire de Mathématiques is Unité Associé au CNRS No. 168.

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