Critical Behavior in (2 + 1)-Dimensional QCD

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QCD in 2+1 dimensions with N species of fermions, analyzed in the $1/N$ expansion, is shown to exhibit a chiral and confinement phase transition as N approaches a critical value N_c. For $N > N_c$, the theory is chirally symmetric and deconfined. For $N < N_c$, chiral symmetry is broken and confinement sets in at momentum scales less than the fermion mass. The reliability of the $1/N$ expansion when $N \cong N_c$ is discussed, and the behavior of this theory is contrasted with QCD in 3+1 dimensions.

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In a recent Letter,[1] arguments were presented that in (2+1)-dimensional QED (QED3) with N species of fermions, the spontaneous generation of a parity-conserving fermion mass will take place if and only if $N < N_c$, where N_c is some critical value. A global "chiral" symmetry is then spontaneously broken by the appearance of the mass. Although this conclusion was reached working to first order in a $1/N$ expansion, subsequent studies have provided further evidence that the existence of this critical point is a genuine feature of QED3 and that the $1/N$ expansion can be reliably used to describe it. First of all, computer simulations using lattice techniques have concluded that an N_c indeed exists and that $3 < N_c < 4$.[2,3] Second, a computation of the next-order terms in the $1/N$ expansion[4] have shown that for $N \sim N_c$, they are relatively small and do not qualitatively change the lowest-order result. The theory appears to undergo a chiral phase transition, triggered by the screening effect of the fermions.[5]

More generally, it is possible to consider the spontaneous generation of both a chiral-symmetry-breaking fermion mass and a parity-violating fermion mass with a corresponding Chern-Simons term for the gauge field. In QED3, this has been considered within the context of the $1/N$ expansion.[6,7] It was shown that to leading order in the expansion, it is only chiral symmetry that breaks spontaneously.[8–10]

In this Letter we summarize a study of critical behavior in a (2+1)-dimensional Yang-Mills theory (QCD3) with N species of fermions. Making use of a $1/N$ expansion, we argue that a single critical point N_c exists, above which the theory is chirally symmetry and deconfined, due to the screening effect of the fermions and below which confinement and chiral-symmetry breaking sets in. Previous studies have concluded that QCD3 without dynamical fermions is a confining theory.[11] Assuming these results to be correct, the onset of confinement for $N < N_c$ is an immediate consequence of the appearance of a dynamical fermion mass. The new phenomenon to be demonstrated here is the disappearance of confinement for $N > N_c$.

The possibility of the spontaneous generation of a parity-violating fermion mass and Chern-Simons gauge term will not be considered here, except to note that just as in QED3, it will not happen to leading order in the $1/N$ expansion.

We consider an arbitrary gauge group G with N four-component Dirac fermions transforming according to some representation R. The only intrinsic dimensionful parameter in the theory is $\alpha = Ng^2/8$, where g is the gauge coupling constant. Being super-renormalizable, the theory is rapidly damped at momentum scales above α.

The theory has a U(2N) global, "chiral" symmetry. We consider the possibility that this symmetry is spontaneously broken to U(N)×U(N) through the appearance of a parity-conserving dynamical mass $\Sigma(p)$ for each of the four-component Dirac spinors. We estimate the critical value N_c for spontaneous chiral-symmetry breaking by solving the gap equation to lowest order in $1/N$ (ladder approximation). This part of the analysis goes exactly as in QED3 and we simply report the results. To this order, the gauge-invariant critical value N_c for the onset of chiral-symmetry breaking is

$$N_c = \tfrac{4}{3}\, 32 C_2(R)/T(R)\pi^2 , \qquad (1)$$

where $C_2(R)$ is the quadratic Casimir invariant for representation R and $T(R) = C_2(R)d(R)/r$, with $d(R)$ the dimension of the representation R and r the dimension of the group. If the gauge group is taken to be SU(M) and the fermions are in the fundamental representation, $C_2(R) = (M^2-1)/2M$ and $T(R) = \frac{1}{2}$. Thus $N_c = 64/\pi^2$ for SU(2). The factor of $\frac{4}{3}$ in Eq. (1) arises from a careful treatment of wave-function renormalization.[4] If the next-order corrections in the $1/N$ expansion behave roughly as they do in QED3,[4] they will decrease these estimates of N_c by less than 20%.

The dynamical fermion mass $\Sigma(p)$ falls monotonically with p from some initial value $\Sigma(0)$ whose form for $N < N_c$ is

$$\Sigma(0) = C\alpha \exp[-2\pi/(N_c/N - 1)^{1/2}] . \qquad (2)$$

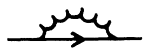

FIG. 1. Fermion self-energy order $1/N$. The wavy line is the zeroth-order gluon propagator.

FIG. 2. Fermion-gluon vertex to order $1/N$, restricted to graphs that can give rise to $\ln(\alpha)$ terms as $\alpha \to \infty$.

The constant C is of order unity so that $\Sigma(0) \ll \alpha$ for N near N_c. The exponential factor is insensitive to the details of both the ultraviolet regime $p > \alpha$, and the infrared, nonlinear regime $p < \Sigma(0)$.[1]

To discuss confinement, we first imagine that $N \gg N_c$, so that the fermions remain massless and the $1/N$ expansion can be expected to be reliable. To zeroth order in $1/N$, the gluon propagator is $D_{\mu\nu}(q) = (Q_{\mu\nu}/q^2) \times [1+\Pi(q)]$, where $Q_{\mu\nu} = g_{\mu\nu} - q_\mu q_\nu/q^2$ in Landau gauge, and $\Pi(q) = T(R)\alpha/q$. For $q \ll \alpha$, the interaction between two charges is proportional to $g^2 D_{\mu\nu}(q) \sim [1/NT(R)]Q_{\mu\nu}/q$. In this limit the form of the interaction is Coulombic $[V(r) \sim 1/r]$ and the coupling strength is the dimensionless parameter $1/NT(R)$. Thus, to zeroth order, the effective infrared theory is scale invariant and nonconfining. The scale invariance can also be seen by constructing the dilation current of the theory. Its divergence is nonvanishing classically, but a quantum correction develops at zeroth order in the $1/N$ expansion, which removes the classical piece in the limit $\alpha \to \infty$.[12]

We next show that this remains true to all orders in the $1/N$ expansion. To be more precise, we show that for physical quantities, the $1/N$ expansion for $q \ll \alpha$ is not modified by the appearance of $\ln(\alpha/q)$ terms. Since the infrared power counting of this theory is essentially that of a renormalizable theory,[12,13] it is straightforward to see that the only possible source of such terms is the renormalization of the effective low-energy expansion parameter $1/N$. It will be shown that to any order in the $1/N$ expansion, there are no $\ln(\alpha/q)$ terms in this renormalization. Thus the effective low-energy quantum theory is scale invariant (the effective β function vanishes) and the long-range forces remain of the form $1/q$ and nonconfining to all orders in the $1/N$ expansion. It has already been argued[13] that this is a feature of QED3 using the Abelian Ward identity. Here, we show by a different argument that it is a low-energy feature of all massless three-dimensional gauge theories in the $1/N$ expansion.

To isolate the renormalization corrections to $1/N$, one can consider, for example, the combination of quantities $Z_3^{1/2}Z_2/Z_1$, where Z_3 (Z_2) denotes gluon (fermion) wave-function renormalization and Z_1 denotes gluon-fermion vertex renormalization. Other, gauge-equivalent combinations could also be considered. The Z's must be evaluated at some reference scale $\mu \ll \alpha$ appropriate to describe the low-momentum theory.

Consider a contribution to any one of them computed to some order in the $1/N$ expansion. A simple power-counting analysis in the effective infrared theory, using the fact that $D_{\mu\nu}(q) \sim 1/q$, shows that for Z_2 and Z_1, the degree of divergence d is given by $d = N_3 + N_4$, where N_3 and N_4 are the number of three-gluon and four-gluon vertices in the graph. Now if $d > 0$, the scale μ may be set to zero without encountering any infrared divergences, and the apparent ultraviolet divergences in this effective infrared theory will be cut off at α. The factors of α will cancel in the dimensionless Z_1 and Z_2 and no $\ln(\alpha/\mu)$ terms are possible because of the absence of infrared divergences as $\mu \to 0$. It follows that the only graphs that can give rise to $\ln(\alpha)$ terms are those without direct gluon self-interactions. To order $1/N$, they are shown in Figs. 1 and 2.

A corresponding power-counting analysis for the gluon self-energy gives the naive expression $d = N_3 + N_4 + 1$, which is then decreased to $d = N_3 + N_4 - 1$ taking into account invariance. Therefore, with no direct gluon self-interactions, $d = -1$, meaning that the result is proportional to $1/q$, where q is the external momentum. Multiplying this by the q^2 that arises from gauge invariance, the result has exactly the behavior required to survive as a leading term in the low-energy expansion. By contrast, any contributions with $d > -1$ will lead to nonleading terms in the low-energy expansion. It follows that only graphs containing no direct gluon self-interactions will contribute. The surviving contributions of order $1/N$ are shown in Fig. 3.

For these graphs, however, their overall ultraviolet finiteness in the effective infrared theory means that there are no factors of $\ln(\alpha)$ arising from the graphs as a whole. The only $\ln(\alpha)$ terms come from the vertex and fermion self-energy subgraphs. Inspection reveals that there is a one-to-one correspondence between them and the self-energy and vertex graphs of Figs. 1 and 2. The counting factors are the same and they pair up with opposite signs due to the extra fermion loops in Fig. 3. Thus the various $\ln(\alpha)$ terms in $Z_3^{1/2}Z_2/Z_1$ cancel, showing that to this order, the effective β function vanishes and the theory remains scale invariant. Notice that this result does not make direct use of the Ward identities of the theory; instead it relies on a cancellation that

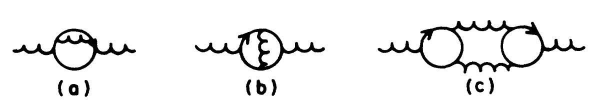

FIG. 3. Gluon self-energy to order $1/N$, restricted to graphs that can give rise to $\ln(\alpha)$ terms as $\alpha \to \infty$.

operates in the $1/N$ expansion, but not in the loop expansion, and applies to both Abelian and non-Abelian theories.

It is straightforward to generalize this result to all orders in the $1/N$ expansion. Consider first contributions not involving Fadeev-Popov (FP) loops. To any order, it can be seen that the only objects that survive as leading terms in the low-energy expansion and furthermore could give rise to $\ln(\alpha)$ terms are fermion self-energy graphs and gluon-fermion vertex graphs containing no direct gluon self-couplings. Having established this, it is easy to demonstrate that the above cancellation mechanism will always take place in the combination of quantities that renormalize $1/N$. For every Z_1 and Z_2 associated with the external fermion, there will be a corresponding and canceling Z_1 and Z_2 associated with the fermion loops in Z_3. A detailed demonstration involves a conventional disentangling of overlapping divergences inside Z_3.

A gauge-equivalent construction of the effective β function could have begun with the three-gluon or four-gluon vertex instead of the gluon-fermion vertex. However, because graphs with direct gluon self-interactions cannot contribute $\ln(\alpha)$ terms in the effective low-energy theory, it is not difficult to see that either of these approaches will reduce immediately to the gluon-fermion vertex case discussed above.

The Fadeev-Popov (FP) ghost requires a separate treatment. From the above discussion, it should be clear that the FP ghost loops as a whole do not lead to $\ln(\alpha)$ factors. The only way in which $\ln(\alpha)$ terms could enter associated with the FP ghosts is through the ghost self-energy (Z_{2g}) or the ghost-gluon vertex (Z_{1g}). The degree of divergence of both these quantities in the effective infrared theory is $d=N_3+N_4$. Just like the fermion self-energy and the fermion-gluon vertex, only contributions without direct gluon self-interactions can give rise to $\ln(\alpha)$ terms. The contributions of order $1/N$ are analogous to Figs. 1 and 2.

Now Z_{2g} and Z_{1g} do not stand alone in a physical amplitude. They appear with the gluon self-energy (Z_3) in the combination $Z_3^{1/2}Z_{2g}/Z_{1g}$, renormalizing $1/N$ and giving yet another gauge-equivalent definition of the β function of the effective low-energy theory. Since the earlier definition led to the cancellation of the $\ln(\alpha)$ terms and the vanishing of the effective β function, however, the same must happen here. This consistency is ensured by the Slavnov-Taylor identities of the theory.[14] By evaluating them at a scale $\mu \ll \alpha$, the following identity among the above renormalization constants of the effective infrared theory is obtained:

$$Z_{2g}/Z_{1g}=Z_2/Z_1 . \quad (3)$$

Thus, since the right-hand side, multiplied by $Z_3^{1/2}$, is finite to all orders in $1/N$ in the limit $\alpha \to \infty$, so too is the left-hand side, multiplied by $Z_3^{1/2}$. While the former cancellation is followed from the straightforward analysis described earlier, the latter cancellation is not quite so easy to prove directly. We have, however, checked directly that the cancellation takes place at order $1/N$.

To summarize, we have shown that to all orders in the $1/N$ expansion, the effective infrared ($q \ll \alpha$) quantum theory becomes scale invariant in the sense that there are no $\ln(\alpha)$ corrections to the renormalization of $1/N$. The effective infrared β function of the theory vanishes. This absence of $1/N$ scale anomalies ensures that the divergence of the dilation current, which vanishes to zeroth order in $1/N$ for $q \ll \alpha$, retains this property to all orders.[12] Long-range forces remain Coulombic ($1/q$) to all orders in $1/N$, and with N larger than N_c so that the fermions remain massless, this behavior will persist to infinitely large distances. We conclude that to all orders in $1/N$, the screening effect of the large number N of fermions deconfines the theory.

We have of course only considered the structure of the leading terms in the low-energy expansion of QCD3. The analysis should be extended to include nonleading terms in the expansion in q/α, perhaps organized into a tower of higher-dimension operators. We have also not considered the possible role of "non-$1/N$" structures, nonanalytic in $1/N$ and therefore inaccessible through the $1/N$ expansion.

The above discussion can be generalized to include additional massless fermions, either in different representations or in the same representation as the N fermions doing the screening. The Slavnov-Taylor identities will again ensure scale invariance for $q \ll \alpha$, to all orders in $1/N$, and therefore the deconfinement of all the gauge and matter fields. The discussion can also be generalized to include massive matter fields. Consider, for example, the addition of a single Dirac fermion F of mass M in some representation of G. It can be shown that to any order in $1/N$, the long-range ($q \ll \alpha, M$) interaction between, say, F and $\bar{F}$ remains Coulombic.

Now suppose that N is decreased to just below N_c. Spontaneous breaking will lead to a fermion mass $\Sigma(0)$ [Eq. (2)]. For N near N_c, $\Sigma(0) \ll \alpha$ and there will be a range of momenta where the screening effect described above will remain active. Assuming that the $1/N$ expansion is reliable here, the quarks and gluons will experience only relatively weak Coulombic ($1/q$) forces for $\alpha \gg q \gg \Sigma(0)$, and exhibit no evidence of confinement. It has been argued[1] that this is the dominant momentum range governing the onset of chiral-symmetry breaking and leading to the exponential factor in Eq. (2). Thus, since the driving force remains Coulombic to all orders in $1/N$ in this range, the exponential (infinite-order) character of the chiral phase transition will also be true to all orders.

For $q \ll \Sigma(0)$, however, the fermions will freeze out and the effective theory will be QCD3 without fermions. Its effective coupling constant, found by integrating out the fermions, is $g_{\text{eff}}^2=O(g^2\Sigma(0)/\alpha)=O(\Sigma(0)/N)$. Pure

QCD3 is a theory in which the loop expansion rapidly breaks down due to infrared divergences. It is believed to confine,[11] with the confinement scale given by the effective coupling $g^2_{\rm eff}$. For N near N_c, this scale will be much less than α. The long-range potential will be linear, as opposed to logarithmic as in QED3,[5] with coefficient (string tension) proportional to $g^4_{\rm eff} \sim \Sigma^2(0)/N^2$. The string tension will vanish according to Eq. (2) as N approaches N_c from below.

As N decreases farther below N_c, the confinement scale will move up and become of order α. For these low values of N, of course, there is no reason to expect the $1/N$ expansion to be convergent. Nevertheless, if the expansion is convergent for N in the vicinity of N_c (as it seems to be in QED3), then the chiral and confinement critical point N_c described here will be a genuine feature of QCD3. As in QED3, both lattice simulations and the computation of higher-order corrections in $1/N$ will be important to confirm this result.

SU(2) gauge theories in 2+1 dimensions are of interest for understanding the properties of strongly coupled electronic systems in statistical mechanics.[15] While the fermionic screening out of confinement for $N > N_c$ may not be directly relevant for the models of physical interest, it is a feature of the general class of models and may be of use in future studies.

Finally, we make a few comparative remarks about the role of fermions in QCD3 and QCD4. In the loop expansion of QCD4 with N_f fermions, there is a critical number N_{fc} beyond which asymptotic freedom is lost. If the coupling is weak at some momentum scale, it becomes weaker as the momentum is decreased, as long as the fermions are light enough to remain active. If they are massless, confinement is lost. What happens at high momentum depends on where new physics enters. If the growing coupling is not to trigger large dynamical fermion masses, the new physics must enter at an appropriate point and cut off the growth. At any rate, this discussion, being based in the loop expansion, is restricted to weak coupling. A $1/N_f$ expansion is not directly useful for going beyond weak coupling, since it leads to tachyonic poles at zeroth order.

QCD3, by contrast, is a super-renormalizable theory, and therefore superasymptotically free for any number N of fermions. Increasing this number simply screens the infrared behavior. The $1/N$ expansion contains no pathologies, and as we have shown, leads to the identification of $1/N$ as the effective expansion parameter in the infrared regime. The loss of confinement for $N > N_c$, leaves the soft ultraviolet behavior intact and the theory is, in effect, weakly coupled at all momentum scales.

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