Nonperturbative Solution of the Ising Model on a Random Surface

David J. Gross and Alexander A. Migdal
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
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The two-matrix-model representation of the Ising model on a random surface is solved exactly to all orders in the genus expansion. The partition function obeys a fourth-order nonlinear differential equation as a function of the string coupling constant. This equation differs from that derived for the $k=3$ multicritical one-matrix model, thus disproving that this model describes the Ising model. A similar equation is derived for the Yang-Lee edge singularity on a random surface, and is shown to agree with the $k=3$ multicritical one-matrix model.

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Recently dramatic progress has been made in the study of random surfaces or two-dimensional gravity.[1-3] The one-matrix model, which yields a discretized definition of the sum over random two-dimensional surfaces, has been solved nonperturbatively. This model, originally developed to reproduce the genus expansion of the partition function for two-dimensional gravity, turns out to yield a differential equation that sums the expansion. In the case of pure two-dimensional gravity, with no matter coupled, it was shown that the specific heat $f(t)$ satisfies the Painleve equation $f^2-\frac{1}{3}\ddot{f}=t$, where t is essentially the inverse string coupling constant. This is of great interest since this model provides the simplest example of string theory, albeit one with a finite number of degrees of freedom, where nonperturbative methods are sorely lacking.

In this work it was also claimed that one could study with the same techniques the complete series of unitary theories of conformally invariant matter coupled to two-dimensional gravity, by studying the multicritical points of the one-matrix model. This claim was based on Kazakov's discovery of these multicritical points and his argument that they described the unitary series.[4] This argument was largely based on the coincidence that the string anomalous dimension calculated for the kth multicritical point was given by $\gamma_s=-1/k$, in agreement with the Knizhnik-Polyakov-Zamolodchikov[5] (KPZ) calculation of γ_s for the unitary conformal theory with central charge $c=1-6/k(k+1)$. With this correspondence one could derive differential equations for the specific heat of these models. For example, in the case of $k=3$, which was identified with the Ising model on a random surface, the specific heat obeyed the equation

$$f^3-ff''-\tfrac{1}{2}(f')^2+\tfrac{1}{10}f^{(4)}=t\,. \tag{1}$$

However, recently some doubts as to the validity of this identification have appeared. They include the following.

(1) The power-series expansion of $F(t)$ in inverse powers of $t^{2+1/k}$ yields the partition function on surfaces of given genus. These must be positive if the theory is unitary. We have used the differential equations to calculate these to high order,[6] with the result that for $k>2$ they eventually, for some genus, turn negative. In particular for $k=3$, the purported Ising model, this occurs first at genus 6. This is a strong argument that the multicritical points of the one-matrix model are nonunitary, and certainly cannot coincide with the unitary series coupled to two-dimensional gravity. This is perhaps not too surprising given that the Kazakov multicritical points require triangulations with negative weights. Furthermore, the value of γ_s, or c for that matter, by itself cannot determine the model uniquely.

(2) Staudacher[7] has proposed identifying the $k=3$ multicritical point of the one-matrix model with the Yang-Lee edge singularity on a random surface. He argues, for surfaces with spherical topology, that the nonunitary theory of an Ising model above its critical temperature in a purely imaginary magnetic field coincides, in the limit of infinite temperature, with a hard-dimer problem that can be mapped onto a ϕ^6-field theory. The critical behavior of this theory is precisely that of Kazakov's $k=3$ multicritical point. This suggests that the correct identification of the $k=3$ multicritical theory of the Yang-Lee edge singularity coupled to gravity, for which Cardy has argued,[8] $c=-\frac{22}{5}$.

(3) Witten has recently reproduced the correlation functions, calculated for these models in Ref. 1, in terms of a topological theory of gravity.[9] From the point of view of this formulation there is no reason for unitarity at all.

(4) Polyakov has argued that these models can be identified with the nonunitary (for $k>2$), ($q=2$, $p=2k-1$), minimal model coupled to two-dimensional gravity, and that the KPZ calculation of γ_s fails due to the presence of negative-dimension operators.[10] A corrected calculation yields $\gamma_s=-1/k$ in accord with the one-matrix model.[4]

It is therefore important to try to directly solve the Ising model to all orders and compare with the $k=3$ multicritical point. In this Letter this will be done with the result that, indeed, the two are not equivalent for vanishing magnetic field, but coincide for imaginary field at the Yang-Lee singularity.

The Ising model on a random surface with the topology of a sphere was formulated by Bershadsky and Migdal,[11] solved in the case of zero field by Kazakov,[12] and in the case of an external magnetic field by Boulatov and Kazakov.[13] Here we shall extend these results to all orders, summing over surface with arbitrary topology.

Let us recall the formulation of the two-matrix problem that generates Ising spins in an external magnetic field H on a random surface. Consider a model of two N-by-N Hermitian matrices X and Y, with the following partition function:

$$Z_N=\int DX\,DY\,e^{-\beta W(X,Y)}=\int\prod_i dx_i\,dy_i\,\Delta_N(x_i)\Delta_N(y_i)e^{-\beta W(x,y)}\,, \tag{2}$$

$$W(X,Y)=\mathrm{Tr}[(X^2+Y^2-2cXY)-g(e^H X^4+e^{-H}Y^4)]\,,$$

where $\Delta_N(x_i)=\prod_{i<j=1}^N(x_i-x_j)^2$. It is not difficult to show that the free energy of this model $F=\ln Z_N$ is equal to a sum over random surfaces with Ising spins attached. In the limit of large N only planar graphs survive in the g expansion, each vertex corresponds to an X (or Y), which represents an up spin (or a down spin). The equivalence of the two models can easily be established.[4] Keeping higher-order terms in $(1/N)^{2h}$ can reproduce the Ising model on a random surface with h handles.

This model was solved as $N\to\infty$, i.e., the Ising model on a random sphere, by the method of orthogonal polynomials, which was first exploited by Mehta to solve the two-matrix problem.[14] One defines orthogonal polynomials $P_n^+(x)$ and $P_n^-(y)$ with respect to the measure in (2)

$$\int dx\,dy\,e^{-\beta W(x,y)}P_n^+(x)P_m^-(y)=h_n\delta_{nm}\,. \tag{3}$$

These are useful since the Van der Monde determinants Δ_N can be expanded as products of these polynomials, the integrals performed and Z_N can be expressed as $\prod_i h_i$.[14] To solve the model we need to calculate the h_i.

From the definition of the orthogonal polynomials it follows that they obey a three-term recursion relation

$$xP_n^{\pm}(x)=P_{n+1}^{\pm}(x)+R_n^{\pm}P_{n-1}^{\pm}(x)+S_n^{\pm}P_{n-3}^{\pm}(x)\,. \tag{4}$$

(Note that the fact that this recursion relation has three terms is a consequence of the fact that W is quartic. If W were of higher order, so would be the recursion relation. This complicates the exploration of the multicritical points of this model.) If the magnetic field vanishes, then $P^+=P^-$, $R^+=R^-$, $S^+=S^-$, otherwise they differ. Equations for the coefficients $P^{\pm},R^{\pm},S^{\pm}$ can easily be derived using properties of the orthogonal polynomials; for example,

$$\int dx\,dy\,e^{-\beta W(x,y)}\frac{dP_{n-1}^{+}(x)}{dx}P_n^-(y)=0\,,$$

$$\int dx\,dy\,e^{-\beta W(x,y)}\frac{d}{dx}[P_n^+(x)-x^n]P_n^-(y)=0\,.$$

One finds

$$\begin{aligned}
cR_n^{\mp}&=y_n[1+2ge^{\mp H}(R_{n+1}^{\pm}+R_n^{\pm}+R_{n-1}^{\pm})]\,,\\
cy_n+\frac{n}{2\beta}&=R_n^{\pm}+2ge^{\pm H}\\
&\quad\times[(S_{n+2}^{\pm}+S_{n+1}^{\pm}+S_n^{\pm})\\
&\quad+R_n^{\pm}(R_{n+1}^{\pm}+R_n^{\pm}+R_{n-1}^{\pm})]\,,\\
cS_{n+1}^{\mp}&=2ge^{\mp H}y_{n+1}y_ny_{n-1}\,,
\end{aligned}\tag{5}$$

where we have defined $y_n\equiv h_n/h_{n-1}$. We must solve these equations for y_n, from which we can calculate $F_N=\sum_{n=1}^N(N-n)\ln y_n$.

In the large-N limit we can replace n/β by a continuous variable x, and replace $y_n\to y(x)$, $R_n^{\pm}\to(x)$, $S_{n+1}^{\pm}\to S^{\pm}(x)$ (note the shift by one unit). The free energy is then given by $F_N(\beta)\propto\beta^2\int_0^X dx(X-x)\ln y(x)$. The scaling laws arise from the singular behavior of $y(x)$ near the point $x=1$, when β/N equals its critical value 1. If we were to neglect the difference between, say, y_n and y_{n+1}, we would then derive equations that would yield the model on the sphere. However, as in the treatment of the one-matrix model, we should keep the corrections of order $1/N$ in the expansion of

$$y_{n+1}=y(x)+\frac{1}{\beta}y'(x)+\frac{1}{2}\left(\frac{1}{\beta}\right)^2y''(x)+\frac{1}{6}\left(\frac{1}{\beta}\right)^3y'''(x)+\frac{1}{24}\left(\frac{1}{\beta}\right)^4y^{(4)}(x)+\cdots\,,$$

if we are interested in higher-genus surfaces. (It will not be necessary to include terms with higher than four derivatives.) They will be enhanced by inverse powers of $\beta/N-1$, which goes to zero at the critical point of the theory.

So we expand

$$\begin{aligned}
cR^{\mp}&=y\left[1+6ge^{\pm H}\left(R^{\pm}+\frac{1}{3\beta^2}R^{\pm\prime\prime}+\frac{1}{36\beta^4}R^{\pm(4)}\right)\right]\,,\\
cS^{\mp}&=2ge^{\pm H}y\left(y^2-\frac{1}{\beta^2}y'^2+\frac{1}{\beta^2}yy''+\frac{1}{12\beta^4}yy^{(4)}\right)\,,\\
cy+\frac{x}{2}&=R^{\pm}+6ge^{\pm H}\left[S^{\pm}+\frac{1}{3\beta^2}S^{\pm\prime\prime}+\frac{1}{36\beta^4}S^{\pm(4)}+R^{\pm}\left(R^{\pm}+\frac{1}{3\beta^2}R^{\pm\prime\prime}+\frac{1}{36\beta^4}R^{\pm(4)}\right)\right]\,.
\end{aligned}\tag{6}$$

We must now adjust the parmeters g and c so that the surface and the spins become critical, i.e., that the renormalized cosmological constant vanish and the spins are not frozen (massive). In general, for arbitrary H it is cumbersome to solve for these critical values. However, in two cases, it is relatively easy: that of vanishing magnetic field and that of $H=i\pi/2$.

First, the case of vanishing field, where (6) reduces to three coupled differential equations. In this case $g_{\text{critical}}=-\frac{5}{72}$ and $c_{\text{critical}}=\frac{1}{4}$. In other words, for these critical values, we find that as $x\to 1$, $y(x)\to\frac{3}{5}(1-f)$, $R\to\frac{6}{5}(1-r)$, $S\to-\frac{3}{25}(1-s)$, and that as $x\to 1$, y vanishes according to $\frac{2}{5}y^3=1-x$. This yields in the spherical limit $F\sim t^{2+1/3}$, where we have defined the scaling variable $t\equiv(1-x)\beta^{6/7}$, corresponding to a *string anaomalous dimension*, $\gamma_s=-\frac{1}{3}$.

It is now a straightforward matter to examine (6) in the vicinity of $x\sim 1$. These three equations can be reduced to a single equation for $f(t)=\ddot{F}(t)$:

$$f^3-\tfrac{3}{2}ff''-\tfrac{3}{4}(f')^2+\tfrac{1}{10}f^{(4)}=t\,. \tag{7}$$

We have rescaled $t\to(\frac{2}{5})^{1/7}t$, $f\to(\frac{2}{5})^{-2/7}f$, $F\to F$, so as to set the coefficient of f^3 equal to 1. This corresponds to a renormalization of the string coupling. If we allow ourselves to rescale the free energy as well we could rewrite this equation in a form similar to (1), since given the structure of the differential equation one can always choose the coefficient of f^3 and the coefficient of $-ff''$ to be 1, as

$$f^3-ff''-\tfrac{1}{2}(f')^2+\tfrac{2}{27}f^{(4)}=t\,. \tag{8}$$

The disagreement of the coefficients of (7) and (1) definitely establishes that the Ising model differs from the $k=3$ multicritical one-matrix model. For example, $F_{\text{Ising}}=\frac{9}{20}t^{2+1/3}+\frac{1}{12}\ln(t)+\cdots$, whereas $F_{k=3}=\frac{9}{20}t^{2+1/3}+\frac{1}{18}\ln(t)+\cdots$. Does this shift of the coefficients help improve the positivity of the partition function? We have calculated, using the recursion relation that follows from (7) or equivalently (8), the expansion of $f(t)$ in inverse powers of $t^{7/3}$, which is the string coupling constant. One finds that all the terms are positive as long as the coefficient of $f^{(4)}$, in (8), is smaller than $\frac{1}{12}$. In the case of the Ising model the coefficient is $\frac{2}{27}<\frac{1}{12}$, so the series is positive; in the case of the $k=3$ one-matrix model, $\frac{1}{10}>\frac{1}{12}$ so the series becomes negative at high genus.

The fact that the two equations for the specific heat in the two models differ so slightly suggests that the $k=3$ multicritical one-matrix model is some kind of small, nonunitary, perturbation of the Ising model. This is in accord with Staudacher's contention[7] that the $k=3$ multicritical one-matrix model coincides with the hard-dimer problem on a random surface, which coincides with the Ising model for $H=i\pi/2$. In this case the critical value of c is equal to 1, and the critical value of g vanishes. This means, in Ising-model language, that the temperature is infinite. Let us calculate the partition function to all orders for this case.

The $H=i\pi/2$ case is somewhat tricky, since $g_{\text{critical}}=0$. We therefore choose to shift H slightly, taking $H=i(\pi/2-\epsilon)$ and keeping ϵ finite till the end of the calculation. Then one finds that $g_{\text{critical}}=-\frac{2}{225}\epsilon^3$, and that as $x\to 1$, $y(x)\to(15/4\epsilon^2)(1-f)$, $S^{\pm}\to(15/16\epsilon^3)\times e^{\mp H}(1-s^{\pm})$. We now have to reduce the six equations in (6), near $x\sim 1$ and $\epsilon\sim 0$, to an equation for $f(x)$. This is straightforward, although tedious, finally giving in terms of the scaling variable (no rescaling is necessary here) $f^3-ff''-\frac{1}{2}(f')^2+\frac{1}{10}f^{(4)}=t$. This is in precise agreement with Eq. (1), for the $k=3$ multicritical one-matrix model. This result confirms Staudacher's conjecture to all orders in the genus expansion.

In the case of the standard Ising model on a regular lattice there is only one critical theory for nonvanishing (imaginary) magnetic field—the Yang-Lee edge singularity.[15] As soon as one turns on the magnetic field the system jumps from one to the other. We would expect that this should be the case for the matrix-model representation of Ising spins coupled to gravity. To verify this we solved, using the symbolic-computation package MATHEMATICA, Eqs. (6) for arbitrary field H. We obtained the following parametric solution that interpolates between the Ising model ($\alpha=0$) and the Yang-Lee edge singularity at infinite temperature, the hard-dimer model ($\alpha=\pi/2$),

$$c=\frac{4}{|D|},\quad e^H=\frac{D^*}{D}e^{i\alpha},\quad g=-\frac{160\cos^3(\alpha)}{9|D|^2},\quad D=5e^{2i\alpha}+10+e^{-2i\alpha},\quad y=\frac{3(1-f)|D|}{80\cos^2(\alpha)}, \tag{9}$$

$$R^+=Dz\left[\frac{3(3-2f+z)}{40(1+z)^2}+\frac{(3f^2-f'')}{20(-1+z^2)}+\frac{(1+3z)(3f^3-2f'^2)}{20(-1+z)^3}+\frac{-24(1+7z+4z^2)ff''+(3+18z+11z^2)f^{(4)}}{240(-1+z)^3(1+z)}\right]+\cdots, \tag{10}$$

where $z=e^{-2i\alpha}$. The remaining functions $R^-,S^{\pm}$ are given by similar expressions, which directly follow from (6) and the above form of R^+. All these functions depend continuously on α, so that at first glance the universality of the critical behavior is violated even at the perturbative level. This could occur only if there was a marginal operator, as in the Thirring or Baxter models. However, there is no such operator in the KPZ list of operators for either model. Hence, the α dependence should disappear from the observables. After tedious computation we confirmed this expectation. Indeed, it turned out that specific heat satisfies the $k=3$ multicritical equation for the Ising model at the critical point for arbitrary nonvanishing imaginary magnetic field.

As $\alpha \rightarrow 0$, $z \rightarrow 1$ (i.e., $H \rightarrow 0$) there are singularities in all the above expressions; however, the equation for f remains the same for any finite H. Precisely at $H=0$ there are fewer conditions and the system jumps from one equation to another, in accord with the principles of the theory of critical behavior.

We can conclude from our results, as well as the arguments of Staudacher,[7] that the naive identification of the kth multicritical one-matrix model with the theory in which the unitary conformal series is coupled to two-dimensional gravity is wrong. Instead it should be identified with the Yang-Lee edge singularity.

We have no reason to suspect that the $k=2$ one-matrix model, corresponding as it does to triangulations with purely positive weights and having a positive partition function for all genus, is not unitary. Most likely, however, all the higher multicritical points correspond to nonunitary theories. However, one might conjecture that, since they have the same critical exponents as the unitary models, they might be understood as the coupling of these models to some external complex fields. It would be interesting to explore this possibility.

To this end, as well as for the general development of the nonperturbative two-dimensional gravity, it is imperative to develop methods that would allow us to extend the results of Refs. 1-3 directly to the unitary series. These can be formulated as matrix models and solved on the sphere, but the methods of Refs. 1-3 must be generalized to deal with them beyond lowest order. Work along these lines is in progress.

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Note added.—After completion of this paper we received a paper from E. Brezin, M. Douglas, V. Kazakov, and S. Shenker (to be published) in which the solution of the Ising model in zero magnetic field was presented.

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