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Example of Infinite Statistics

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I give a second-quantized example of infinite statistics (in which any representation of the symmetric group can occur) for identical particles and discuss the properties of the example.

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Since the work of Green[1] in 1953, we have known that the possible statistics of identical particles include para-Bose and para-Fermi statistics of positive integral order p (for $p=1$ these reduce to Bose and Fermi statistics). There is a neat way to describe which representations of the symmetric group occur in each case: For the para-Bose case of order p exactly those representations of the symmetric group with at most p rows in their Young patterns occur, which means that at most p particles can be in an antisymmetric state; for the para-Fermi case exactly those representations with at most p columns occur, which means that at most p particles can be in a symmetric state. The systematic classification of particle statistics in greater than two space dimensions by Doplicher, Haag, and Roberts[2] and the discussion of trilinear commutation relations by Govorkov[3] found only one new case in addition to para-Bose and para-Fermi statistics: "infinite" statistics in which all representations of the symmetric group can occur. These authors did not give an example of infinite statistics. The purpose of this Letter is to give a second-quantized example of infinite statistics and to discuss its properties.

Bose and Fermi statistics are characterized by commutation and anticommutation relations, respectively:

$$[a(k),a^\dagger(l)]_- = \delta_{k,l}\,, \tag{1}$$

$$[a(k),a^\dagger(l)]_+ = \delta_{k,l}\,, \tag{2}$$

where $[A,B]_\pm = AB \pm BA$. Hegstrom made the perceptive suggestion to find a new operator relation which may have properties intermediate to Bose and Fermi statistics by averaging the above relations to get

$$a(k)a^\dagger(l) = \delta_{k,l}\,. \tag{3}$$

Polyakov[4] pointed out that (3) is a special case of the quantum group "q-mutator" relation $a(k)\tilde{a}(l) - q\tilde{a}(l)a(k) = \delta_{k,l}$, where $\tilde{a}$ is $a^\dagger$ for q real and $\tilde{a}_q = a^\dagger_{q^*}$ for q complex.

In this Letter I develop Hegstrom's suggestion. Just as in the Bose and Fermi cases, I assume the existence of a unique vacuum state annihilated by all the annihilators $a(k)$:

$$a(k)|0\rangle = 0\,. \tag{4}$$

These relations allow the calculation of the vacuum-to-vacuum matrix element of any polynomial in the a's and $a^\dagger$'s. To calculate a matrix element which is a monomial in a's and $a^\dagger$'s, consider the rightmost a. If it acts on the vacuum to the right, the matrix element vanishes from (4). If not, it has an $a^\dagger$ immediately to its right. In that case use (3) to replace the pair of operators $aa^\dagger$ by a Kronecker δ and the remaining matrix element has two fewer operators. Continuing this process yields zero, unless the number of a's equals the number of $a^\dagger$'s, in which case the matrix element is a product of Kronecker δ's. This is similar to the calculation for Bose or Fermi operators, except that for infinite statistics the vacuum matrix element of a monomial in a's and $a^\dagger$'s is a single product of Kronecker δ's, so that the calculation is much simpler for infinite statistics than for the Bose or Fermi cases. For polynomials in the operators, add up the results for the monomials of which the polynomial is composed. Note that no statement need be made about aa or $a^\dagger a^\dagger$; in matrix elements these can be evaluated from Eqs. (3) and (4) above. In particular, consider $\langle 0 | a_{i_m} \cdots a_{i_2} a_{i_1} a^\dagger_{j_1} a^\dagger_{j_2} \cdots a^\dagger_{j_n} | 0 \rangle$. Clearly, this matrix element vanishes unless $m=n$ and $i_k = j_k$ for all k. When these conditions are satisfied, the matrix element equals 1. Thus the norm of every monomial in the $a^\dagger$'s acting on $|0\rangle$ is 1. Note that although multiple occupancy of a single quantum state is allowed (in contrast to Fermi statistics), the norm of states such as $[a^\dagger(k)]^n |0\rangle$ remains 1, unlike the case of Bose statistics. This makes it clear that if we choose a polynomial $\mathcal{P}$ in which the creation operators are put in an arbitrary representation of the symmetric group the norm of

$$\mathcal{P}(a_i^\dagger)\,|0\rangle \tag{5}$$

is positive. Thus Hegstroms's operator relation leads to infinite statistics.

Define normal ordering in the usual way: All $a^\dagger$'s must be to the left of all a's. The normal-ordered expansion of a monomial in a's and $a^\dagger$'s is a monomial in which all adjacent $aa^\dagger$ pairs are replaced by the corresponding Kronecker δ's. (An expression such as $aaa^\dagger a^\dagger$ is replaced by a product of two Kronecker δ's, etc.) The a's and $a^\dagger$'s which are already in normal order remain as a factor. Again, for polynomials, add up the monomials.

To construct the operators for the energy, momentum, angular momentum, etc., in terms of the annihilation and creation operators, it suffices to construct a set of number operators, n_i, such that

$$[n_i, a_j]_- = -\delta_{i,j} a_j \,. \tag{6}$$

Then the energy operator, for example, is

$$E = \sum_i \epsilon_i n_i \,, \tag{7}$$

where ϵ_i is the single-particle energy for a noninteracting system or the eigenenergy for an interacting system. For parastatistics, the n_i are bilinear in the a's and $a^\dagger$'s; however, for our example of infinite statistics, they are of infinite degree:

$$n_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i^\dagger a_i a_k + \sum_{k_1,k_2} a^\dagger_{k_1} a^\dagger_{k_2} a_i^\dagger a_i a_{k_2} a_{k_1} + \cdots + \sum_{k_1,k_2,\ldots,k_s} a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_s} a_i^\dagger a_i a_{k_s} \cdots a_{k_2} a_{k_1} + \cdots \,. \tag{8}$$

[As far as I know, this is the first case in which the number operator, Hamiltonian, etc., for a free field are of infinite degree. This probably occurs because (3) defines a quantum group.] In verifying that (6) is valid, the contributions with $\cdots aa$ coming from a given term in n_i cancel against a contribution from the next term in n_i so that the commutator telescopes to give the stated result.

Note that the operator relations and the definition of the number operators transform properly under unity transformations of the a's and $a^\dagger$'s. Any kinematics—nonrelativistic, relativistic, whatever—can be chosen for the energy of the free quanta. The transition operators n_{ij} for the transition from state j to state i are defined in analogy to the number operators n_i: just replace $a_i^\dagger a_i$ by $a_i^\dagger a_j$ in each term in n_i. Despite the infinite series for the number operator and other observables, these operators are no worse behaved than the corresponding (para) Bose and Fermi operators.

To construct fields in position space, treat the subscripts i as momentum indices and use the Fourier transform:

$$\phi(x) = (2\pi)^{-3/2} \int d^3k [2\omega(k)]^{-1/2} [a(k) e^{-i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(-k) e^{i\mathbf{k}\cdot\mathbf{x}}] \,. \tag{9}$$

Here the a's and $a^\dagger$'s have nonrelativistic normalization, but I have chosen relativistic four-vector notation for the exponentials in the Fourier transform. Higher-spin fields can be defined in an analogous way.

To discuss possible locality properties of the infinite statistics field, it is useful to have charged fields with particles and antiparticles. I call the particle annihilation and creation operators b and $b^\dagger$ and the antiparticle operators d and $d^\dagger$. Then the charged scalar field is

$$\phi(x) = (2\pi)^{-3/2} \int d^3k [2\omega(k)]^{-1/2} [b(k) e^{-i\mathbf{k}\cdot\mathbf{x}} + d^\dagger(-k) e^{i\mathbf{k}\cdot\mathbf{x}}] \,. \tag{10}$$

The analog of the Hegstrom relation (3) is

$$b(k) b^\dagger(l) = d(k) d^\dagger(l) = \delta_{k,l} \,, \qquad bd^\dagger = db^\dagger = 0 \,, \tag{11}$$

and the analog of the unique vacuum condition is

$$b(k)\,|0\rangle = d(k)\,|0\rangle \,. \tag{12}$$

The charge operator for this field is

$$\hat{Q} = \sum_{y=b,d} \sum_{n=0}^{\infty} \sum_{l_1,\ldots,l_n} y^\dagger_{l_1} \cdots y^\dagger_{l_n} Q y_{l_n} \cdots y_{l_1} \,, \tag{13}$$

where

$$Q = \sum_k [b^\dagger(k) b(k) - d^\dagger(k) d(k)] \,.$$

Other operators which, at least for the free field, are bilinear in creation and annihilation operators can be made to give the desired commutators with the field by a construction analogous to that just given for Q. Let $\mathcal{O}$ be such an operator. Then define

$$\hat{\mathcal{O}} = \sum_{y=b,d} \sum_{n=0}^{\infty} \sum_{l_1,\ldots,l_n} y^\dagger_{l_1} \cdots y^\dagger_{l_n} \mathcal{O} y_{l_n} \cdots x_{l_1} \,. \tag{14}$$

If $\mathcal{O}=\sum_{p,q}o_{p,q}b^\dagger(p)b(q)$, then

$$[\hat{\mathcal{O}},b^\dagger(l)]_-=\sum_p o_{p,l}b^\dagger(p)\,. \quad (15)$$

To save writing, call the operator $\mathcal{O}$ the "core" and the operator $\hat{\mathcal{O}}=\mathcal{A}(\mathcal{O})$ the corresponding "apple." Then

$$[\mathcal{A}(b^\dagger(k)d^\dagger(l)),d^\dagger(p)]_-=b^\dagger(k)d^\dagger(l)d^\dagger(p)\,. \quad (16)$$

Such trilinear terms ruin the local commutativity of $[\mathcal{A}(j_\mu(x)),\phi(y)]_-$, where $j_\mu(x)=i[\phi^\dagger(x),\overleftrightarrow{\partial}_\mu\phi(x)]_+$. (The expression for j_μ seems to be too large by a factor of 2; however, for infinite statistics it is correct.) I have also verified that $\mathcal{A}(j_\mu(x))$ does not commute with itself at spacelike separation. In view of the connection with quantum groups, this may be connected with the noncommutativity of the coordinates of the spaces on which quantum groups act. Furthermore, $\mathcal{A}(j_\mu(x))$ is also not local in the sense of being a functional of the fields $\phi(x)$ and $\phi^\dagger(x)$ in the neighborhood of x, since $\int d^3l \times b^\dagger(l)b(l)$ is a nonlocal functional of the fields $\phi(x)$ and $\phi^\dagger(x)$. The lack of locality may raise questions about the relativistic version of the theory; however, at least, there is a valid nonrelativistic theory of infinite statistics. For the free field both $j_\mu(x)$ and $\mathcal{A}(j_\mu(x))$ are conserved.

Since the particles obeying infinite statistics do not have a local-field theory, there is no spin-statistics restriction for such particles and they can have any spin. It is amusing to note that, despite the failure of local commutativity, the *TCP* theorem is valid for free infinite-statistics fields. Because of (11) and (12), the vacuum-to-vacuum matrix elements of products of free fields are products of two-point functions; thus it suffices to verify the *TCP* theorem for the two-point function. One need only check that

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle=\langle 0|\phi^\dagger(-y)\phi(-x)|0\rangle\,, \quad (17)$$

$$\langle 0|\phi^\dagger(x)\phi(y)|0\rangle=\langle 0|\phi(-y)\phi^\dagger(-x)|0\rangle\,. \quad (18)$$

The extension to arbitrary spin is routine. The validity of the *TCP* theorem despite the failure of the spin-statistics theorem reminds us of the fact that the *TCP* theorem is weaker than the spin-statistics theorem, as illustrated by Jost's example.[5]

To discuss cluster decomposition properties of free infinite-statistics fields, note that, just as for free Bose or Fermi fields, an arbitrary vacuum matrix element of a product of fields is a sum of products of two-point functions. For simplicity, consider the case of a single neutral scalar field, A. Let

$$F^{(n)}(x_1,\ldots,x_n)=\langle 0|A(x_1)\cdots A(x_n)|0\rangle\,. \quad (19)$$

Then $F^{(n)}=0$, for n odd and for even n

$$F^{(n)}=\sum\langle 0|A(x_1)A(x_i)|0\rangle\cdots\langle 0|A(x_j)A(x_n)|0\rangle\,, \quad (20)$$

where the sum runs over all partitions of $1,\ldots,n$ into pairs of numbers in increasing order with the constraint that the two-point function $\langle 0|A(x_i)A(x_j)|0\rangle$ occurs only if $j-i$ is odd. It follows from this result for the free n-point functions and the spectrum of the two-point function, which is the same as for a free Bose field, that cluster decomposition holds. The proof, which I will give elsewhere, follows the line of Jost,[6] who points out that clustering theorems can be proved without using locality.

The statistical mechanics of particles obeying infinite statistics can be derived in a similar way to the usual derivation of Boltzmann statistics. The partition function Z_N is given by

$$Z_N=\sum_{\text{quantum states}} e^{-\beta H}\,. \quad (21)$$

For a given set of occupation numbers $\{n(p)\}$, with $N=\sum_p n(p)$, there are

$$g\{n(p)\}=N!\Big/\prod_p n(p)! \quad (22)$$

orthogonal quantum states. This is just Boltzmann counting without the Gibbs $1/N!$ factor. In view of this last result, we can call the case of infinite statistics "quantum Boltzmann statistics." Thus, for a nonrelativistic gas which obeys infinite statistics the partition function (in standard notation) is

$$Z_N=[V(mkT/2\pi\hbar^2)^{3/2}]^N\,. \quad (23)$$

Govorkov has suggested[7] that quantum Boltzmann statistics corresponds to the statistics of identical particles with an infinite number of internal degrees of freedom, which is equivalent to the statistics of nonidentical particles since they are distinguishable by their internal states. [The internal symmetry could be SU(∞).] In this case, the increase in entropy which occurs when two samples of the infinite-statistics gas at the same temperature and density are mixed avoids the Gibbs paradox of the entropy of mixing of identical molecules.

To conclude, we have filled a gap in the list of second-quantized particle statistics by giving an example of infinite statistics. I am presently studying the q-mutator relation for $-1\le q\le 1$ to construct a family of theories which interpolate between Bose and Fermi statistics.

I am greatly indebted to Professor Roger Hegstrom for suggesting the relation $aa^\dagger=\delta$. I thank Professor A. M. Polyakov for pointing out that (3) is a special case of the q-mutator relation; Professor A. B. Govorkov for informing me of the results mentioned in Ref. 6; and Professor Peter Freund for suggesting the interpolation in q between -1 and 1. I have benefited from discussions with Professor Rabi Mohapatra and Dr. Ioannis Bakas. This work was supported in part by a grant from the National Science Foundation.

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[6]Jost, Ref. 5, pp. 70 and 71.

[7]A. B. Govorkov (private communication). Govorkov has found a quantization which coincides with infinite statistics from the trilinear commutation relations which he studied in Ref. 3.