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Gravitational Anyons

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We describe some novel effects generated by the gravitational Chern-Simons term similar to those induced by its electromagnetic counterpart. In particular, a structureless particle source generates a gravimagnetic field. Conversely, it causes a spinning source to contribute to the static Newtonian potential, thereby violating the equivalence principle; the potential can even become repulsive.

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The generation of spin in a structureless charged particle by coupling it to the electromagnetic Chern-Simons term (CST) in three spacetime dimensions is by now well understood. This phenomenon, through the associated change of statistics, may even by physically relevant to planar-physics models of high-T_c superconductivity (see, for example, Ref. 1, where earlier references are also given). We show here that the gravitational CST has similar effects: A structureless point mass coupled to it generates a gravimagnetic self-field. In each case, a magnetic field can arise because there are transverse circularly symmetric vectors in two-space. We will also exhibit another common consequence of CST coupling which is the converse of the first: Static vectorial sources, namely dipole current and angular momentum density, respectively, generate Coulomb and Newtonian fields. This represents a violation of the equivalence principle in that inertial and gravitational mass are no longer equal (the latter can even become negative) with a similar implication for charge in the electromagnetic case.

There is also one important difference between the two theories: The gravitational CST is of third-derivative order—one higher than that of the conventional Einstein term, whereas the electromagnetic CST is of first order. Hence, in a derivative expansion of the matter-induced effective action, one can neglect the Maxwell term with respect to the CST, while the gravitational expansion will generally be dominated by the Einstein term, the CST representing a (parity violating) correction. (Of course, the dominance of the respective terms in the resulting field equations depends on whether one considers short- or long-range consequences.) One will therefore have to consider the two terms together here; this is topologically massive gravity.[2] Also, because the gravitational CST alone is conformally invariant, it can only couple to a traceless stress tensor such as that of a null particle. To pursue the massive-point-particle analogy we must in any case include the Einstein term to provide explicit conformal-symmetry breaking. For comparison with the Abelian vector model we will consider only the Abelian, linearized, approximation; this should be reasonable in eventual physical applications. The full nonlinear theory, to which we hope to return, will (unlike pure $D=3$ Einstein theory[3]) be more complicated due to self-energy effects.

Vector CST.—We first briefly review the electromagnetic case, slightly generalized to include the Maxwell term for later comparison with gravity. A charged point particle is minimally coupled to the vector potential A_μ. The field strength is determined by the field equations[2,4]

$$\partial_\mu F^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha A_\beta = -j^\nu, \tag{1a}$$

the corresponding action being

$$I = \int d^3x\left(-\tfrac{1}{4}F^2_{\mu\nu} + \tfrac{1}{2}\mu\epsilon^{\nu\alpha\beta}A_\nu\partial_\alpha A_\beta + A_\mu j^\mu\right) + \tfrac{1}{2}\int m\dot{x}^2 dt. \tag{1b}$$

Here μ has dimension L^{-1}, while charge has the same dimension as $A^\mu \sim L^{-1/2}$, a different convention from the usual anyon one. Our signature is $(-++)$,

$\epsilon^{012}=+1=\epsilon^{12}$. Consider a circularly symmetric static point current j^ν with

$$j^0=e\delta^2(\mathbf{r}), \quad j^i=g\epsilon^{ij}\partial_j\delta^2(\mathbf{r}), \tag{2}$$

where we have included a transverse spatial dipole current of strength g. We use gauges which respect the circular and time-invariance symmetries of the source,

$$A_i=\epsilon^{ij}\partial_j V(r)+\partial_i\Lambda(r), \quad A_0=A_0(r). \tag{3}$$

The longitudinal potential $\partial_i\Lambda$ is irrelevant and remains undetermined by (1). The field equations (1) reduce to

$$\nabla^2(A_0+\mu V)=e\delta^2(\mathbf{r}), \tag{4a}$$

$$\epsilon^{ij}\partial_j[\nabla^2 V+\mu A_0+g\delta^2(\mathbf{r})]=0. \tag{4b}$$

The Coulomb and Yukawa Green's functions

$$-\nabla^2 C=\delta^2(r), \quad (-\nabla^2+\mu^2)Y(r)=\delta^2(\mathbf{r}), \tag{5a}$$

are

$$2\pi C=-\ln r, \quad 2\pi Y=K_0(\mu r). \tag{5b}$$

The Bessel function $K_0(x)$ behaves as $-\ln x$ at the origin and as $x^{-1/2}e^{-x}$ asymptotically. In terms of these, (4) is solved by

$$\mu V=-eC+(e+g\mu)Y, \quad A_0=-(e+g\mu)Y. \tag{6}$$

The pure CST model without a Maxwell term is just the large-μ limit of (1), and the corresponding solution there is

$$V=-e/\mu C(r), \quad A_0=-g/\mu\delta^2(\mathbf{r}). \tag{7}$$

The main effect of dropping of the Maxwell term is that A_μ does not propagate; the (dual) field strength becomes proportional to the current and is therefore confined to regions containing the sources, just as there is no curvature outside the $T_\mu v$ sources in $D=3$ Einstein gravity. Even in this limit, however, the spatial current has given rise to a nonvanishing Coulomb potential A_0 in the interior, $j^i\neq 0$, region. In the extended model's solution (6), this electric field (now of finite range) has strength proportional to $e+g\mu$, so that the current contributes to the static electric charge felt by an external test charge; recall that a minimally coupled point charge moves according to the Lorentz force law irrespective of the gauge field's action. The asymptotic field strength, on the other hand, is purely magnetic and is proportional to e only. Thus the "equivalence" between the electric charges defined by the flux from the Gauss law (4a) and by the "Coulomb" force no longer holds, with or without the Maxwell term.

In pure CST theory, the accompanying alteration of statistics, described by the phase change in any circuit about the particle, is proportional to

$$-e\oint A\cdot dl=e^2/\mu.$$

These results remain valid asymptotically, although modified at finite distance, when the Maxwell term is included. For example, sufficiently close to the particle, where Coulomb and Yukawa potentials coincide, a structureless particle ($g=0$) shows no phase anomaly. Curiously, when the charges are "turned" to have $e+g\mu=0$, the two models have the same exterior solution; the "twist" generated by the CST in the extended model has canceled that present in j^i, and the net result is the same as having a pure charge coupled to CST alone. Asymptotically, the lower-derivative term in the gauge field dominates. In this respect (as well as in having a field-source equality) we will see that the Einstein term alone in gravity is like the CST alone in electromagnetism.

Gravitational CST.— We have noted that the appropriate gravitational model incorporating a CST must include the Einstein term as well, both because the latter is of lower derivative order and because the CST alone can only couple to sources with vanishing T^μ_μ, but not to massive particles. (The pure CST model coupled to a lightlike particle will be treated elsewhere. In that theory, spacetime is locally conformally flat except on the null geodesic where the conformal curvature is concentrated.) The field equations of topologically massive gravity are[2]

$$E^\mu_\nu\equiv\sqrt{-g}G^\mu_\nu+\mu^{-1}C^\mu_\nu=-\kappa^2 T^\mu_\nu, \qquad C^\mu_\nu\equiv\epsilon^{\mu\alpha\beta}D_\alpha(R_{\beta\nu}-\tfrac{1}{4}g_{\beta\nu}R). \tag{8}$$

Unlike pure Einstein gravity in $D=3$, whose action can have either sign,[3] this theory is dynamical, and the overall sign of its action is fixed relative to that of matter in order for it to be nonghost. This implies that the sign of the Einstein term, i.e., of κ^2, must be opposite to the one in $D=4$ gravity, which leads to the negative coefficient of $T^{\mu\nu}\equiv\delta I_{\text{matt}}/\delta g_{\mu\nu}$ in (8). We shall also see that the asymptotic limit of our solutions is that of Einstein theory essentially because the differences due to C^μ_ν depend on μr, and Einstein theory is the $\mu\rightarrow\infty$ limit of our model. We will discuss below the significance of the asymptotic metric with the sign required here. The dimension conventions here are that $\mu\sim L^{-1}$, and $\kappa^2\sim M^{-1}$ is not necessarily related to the $D=4$ Einstein constant. The Cotton (conformal) tensor density C^μ_ν is identically traceless, symmetric, conserved and invariant under a local rescaling of the metric. The system (8) is also different from Einstein gravity in that, like the electrodynamics of (1), it can have nonvanishing curvature (field strength) outside the sources as well.

We now solve the linearized version of (8) for a time-independent circularly symmetric source. For convenience (linearized curvature being gauge invariant), we gauge fix the potential $\kappa h_{\mu\nu}\equiv g_{\mu\nu}-\eta_{\mu\nu}$ to display these symmetries, drop the (irrelevant) longitudinal part of h_{0i}, and choose conformal spatial gauge:

$$h_{0i}=\epsilon_{ij}\partial_j W(r), \quad h_{ij}=\phi(r)\delta_{ij}, \quad h_{00}\equiv n(r). \tag{9}$$

All three physical components (W,ϕ,n) will then be determined. Consider a stationary localized particle,

$$T^0_0=-m\delta^2(\mathbf{r})\,,\quad T_{ij}=0\,,$$
$$T^i_0=-\tfrac{1}{2}\sigma\epsilon^{ij}\partial_j\delta^2(\mathbf{r})=\sigma\delta^2(\mathbf{r})\epsilon^{ij}\partial_j\ln r\,. \tag{10}$$

The momentum density is that of a stationary ($\int T^i_0\,d^2x=0$) spinning ($\int\epsilon^{ij}x^iT^j_0\,d^2r=\sigma$) particle.

With the convention $R^\mu_{\nu\alpha\beta}\sim+\partial_\alpha\Gamma^\mu_{\beta\nu}$, so that $R_{\nu\beta}\sim-\frac{1}{2}\Box h_{\nu\beta}$, the linearized Cotton and Einstein tensors for the metric (9) are

$$C^0_0=\tfrac{1}{2}\nabla^4W\,,\quad C^i_0=-\tfrac{1}{4}\epsilon^{ij}\partial_j\nabla^2(\phi+n)\,,\quad C_{ij}=-\tfrac{1}{2}(\delta_{ij}\nabla^2-\partial^2_{ij})\nabla^2W\,,$$
$$G^0_0=\tfrac{1}{2}\nabla^2\phi\,,\quad G^i_0=-\tfrac{1}{2}\epsilon^{ij}\partial_j\nabla^2W\,,\quad G_{ij}=-\tfrac{1}{2}(\delta_{ij}\nabla^2-\partial^2_{ij})n\,. \tag{11}$$

Both tensors are manifestly conserved; C^μ_ν is also traceless and conformal: $C^\mu_\nu(h_{\alpha\beta}+\eta_{\alpha\beta}\xi(x))=C^\mu_\nu(h_{\alpha\beta})$. The field equations (8) have just three independent components,

$$E^0_0\equiv\tfrac{1}{2}\nabla^2(\phi+\mu^{-1}\nabla^2W)=\kappa m\delta^2(\mathbf{r})\,, \tag{12a}$$

$$E^i_0\equiv-\tfrac{1}{2}\epsilon^{ij}\partial_j\nabla^2[W+(1/2\mu)(\phi+n)]=\tfrac{1}{2}\kappa\sigma\epsilon^{ij}\partial_j\delta^2(\mathbf{r})\,, \tag{12b}$$

$$E_{ij}=-\tfrac{1}{2}(\delta_{ij}\nabla^2-\partial^2_{ij})[n+\mu^{-1}\nabla^2W]=0\,. \tag{12c}$$

The desired solution of (12) is given by

$$W=\mu^{-1}\kappa(m+\mu\sigma)(C-Y)\,,$$
$$\phi=\kappa(m+\mu\sigma)Y-2\kappa mC\,, \tag{13}$$
$$n=\kappa(m+\mu\sigma)Y\,.$$

In the absence of any sources, our time-independent field equations have only the trivial solution. In the full nonlinear theory, it has been shown[5,6] that there are no source-free time-independent solutions unless $\nabla^2W\neq0$; in particular this implies that there are no static ($g_{0i}=0$) solutions. We conjecture that there are no asymptotically flat stationary solutions either, just as in source-free Einstein theory (for any D). These are, however, source-free cosmological solutions; explicit examples are given in Refs. 5-7, but these are not accessible in our linearization about flat space.

Let us next discuss the special case in which $m+\mu\sigma$ vanishes. Then the metric (13) reduces to

$$n=W=0\,,\quad \phi=-2\kappa mC\,, \tag{14}$$

which has the same form as that generated by a structureless spinless point mass in pure (linearized) Einstein gravity, since the field equations essentially reduce to "$(1+\mu^{-1}\nabla\times)$"$(G+\kappa^2T)=0$: The "twist" effects of the CST and of T^i_0 have canceled each other, and there is no net angular momentum. (If σ is thought of as representing the angular momentum of a charged system, $\sigma=e^2/2\pi\mu_V$, this occurs for $e^2/2\pi m=-\mu_V/\mu$. The minus sign here expresses the requirement that the two helicities, whose sign is controlled by that of the respective μ's, be opposite.) If the combination $m+\mu\sigma$ does not vanish (in particular if $\sigma=0$), then neither can W, just as for V in the vector case.

The spacetime represented by the generic solution (13) is asymptotically locally flat. Indeed, near spatial infinity, it has the same form as (the linearization of) the exterior (hence flat) Kerr metric generated by spinning massive sources in pure $D=3$ Einstein theory.[3] There, m was the inertial mass, the gravitational mass vanished (because the Newtonian potential n did), and the "inertial" spin was just the σ part of the coefficient of C in W; the "gravitational" spin vanished as well because its contribution to the linearized geodesic equation is through $\Gamma^i_{0j}\dot{x}^j$ and $\Gamma^i_{0j}\sim h_{0i,j}-h_{0j,i}\sim\epsilon_{ij}\nabla^2W\sim\epsilon_{ij}\nabla^2C\sim0$ (incidentally, Γ^i_{0j} is obviously independent of the longitudinal, gauge, part of h_{0i}). We show next that while the inertial quantities are the same in our theory, the gravitational ones do not vanish. The Newtonian potential is now $(m+\mu\sigma)Y(r)$; since the gravitational mass m_g of a source is defined in terms of the static force it generates on a test particle, m_g is now $m+\mu\sigma$. (The fact that the force is of finite, rather than infinite, range in massive gravity is irrelevant.) Thus the equivalence principle is violated (but for quite different reasons than in $D=3$ Einstein gravity) when there is a nonvanishing spinning T^i_0, just as "charge equivalence" was violated in the electromagnetic system. As noted above, gravitational angular momentum is obtained from the geodesic equation through the term $\Gamma^i_{0j}\dot{x}^j\sim\epsilon^i_j\nabla^2W\dot{x}^j\sim Y\epsilon^i_j\dot{x}^j$, which exhibits a torque (again at finite range) imparted to the test particle; by (13) this quantity is the coefficient $\sigma+\mu^{-1}m$ of Y in W. The σ contribution is due to the source's intrinsic spin, while $\mu^{-1}m$ measures the amount of gravimagnetic field generated by m. In the linearized approximation, the inertial quantities are just the source values, namely m and σ for mass and spin, according to the usual flux-integral definition using the left-hand side of the field equations (12), namely,

$$m_I=\kappa^{-2}\int E^0_0\,d^2x=\tfrac{1}{2}\kappa^{-1}\oint\nabla(\phi+\mu^{-1}\nabla^2W)\cdot d\mathbf{S}=m\,, \tag{15}$$
$$S_I=\kappa^{-2}\int\epsilon^{ij}x^iE^j_0\,d^2x$$
$$=\kappa^{-1}\oint\nabla\left[W+\frac{1}{2\mu}(\phi+n)\right]\cdot d\mathbf{S}=\sigma\,,$$

so that

$$m_g = m_I + \mu S_I, \quad S_g = S_I + \mu^{-1} m = \mu^{-1} m_g. \tag{16}$$

In the full nonlinear theory, the results will be more complicated, because of the gravitational field's contributions. However, the relations between the dressed gravitational and inertial quantities should remain similar. It is also an interesting question whether the special properties of the $m+\mu\sigma=0$ source carry over to the full theory, and whether any static many-particle solutions are permitted there.

We have seen that (as expected) the asymptotic form of the general solution (13) is that of the pure Einstein solution (and the two agree everywhere in form for the special case $m+\mu\sigma=0$). In particular, we found that the spatial metric behaves near infinity as

$$\phi \sim -2\kappa^2 mC = +\kappa^2 m/\pi \ln r, \tag{17a}$$

whereas the corresponding Einstein result[3], was, in our units,

$$\ln(1+\phi) = -\kappa_E^2 m/\pi \ln r. \tag{17b}$$

The left-hand side of (17b) of course linearizes to ϕ. The sign difference between (17a) and (17b) is due to the fact that in Ref. 3 the (*a priori* arbitrary) sign of κ_E^2 was so chosen as to give a normal conical two-geometry with positive angular defect. However, we have no such freedom here and are therefore obliged to accept a negative (linearized) angular defect for a positive mass source. This represents a conical space whose angular range exceeds 2π, a result which (being asymptotic) presumably persists in the full nonlinear solution. Consequently, the source mass can now be arbitrarily large (in contrast with the pure Einstein sign above where there was a maximal allowed value of m) as is physically reasonable for the present, dynamical, system. On the other hand, it is precisely because of our κ^2 sign that the theory describes an attractive Newtonian force [since $K_0'(x)<0$], at least for $m+\mu\sigma>0$—and so in particular for normal $m>0$ sources when σ is absent. "Antigravity" is, however, possible for sufficiently negative $\mu\sigma$.

In conclusion, a structureless massive particle coupled to linearized topologically massive gravity exhibits a gravimagnetic field (proportional to m/μ) to an asymptotic observer, just as a charged particle has a magnetic field $\sim e/\mu$ in the corresponding electromagnetic model.

If the particle has "bare" spin as well (for example because it is also coupled to the electromagnetic CST) then we saw that this spin contributes additively to both the gravitational angular momentum and mass, so that the two contributions can even cancel. We noted further that the equivalence of inertial and gravitational quantities is in general violated, precisely because the gravitational mass and spin as felt by a test particle were "twisted" away from the inertial ones, a phenomenon which permits "antigravity" repulsive Newtonian forces.

We have not discussed here the "anyon" aspects of our solutions, namely the statistics change due to the presence of the h_{0i} potential: The latter enters on the same footing as A_i in its coupling to the particle. Hence (for $\sigma=0$) the anomalous phase here is just obtained by replacing e by κm and so is proportional to $(m\kappa^2)m/\mu$. Insofar as the linearized model treated here is Poincaré invariant, the statistics thus induced should agree with the spin defined as the value of the total system's rotation generator (apart from the particle's orbital contribution) for our solution. This calculation is in progress.

Finally, one may speculate that to the extent that material stresses are relevant in planar systems, these would induce an effective $D=3$ gravitational contribution whose lowest terms in a derivative expansion would generate both Einstein and Chern-Simons terms to realize the mechanism presented here.

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Note added.—Since this paper was submitted, I received a preprint by B. Linet in which the solution (13) for $\sigma=0$ was also obtained, and a preprint by M. E. Ortiz confirming the conjecture in text that the $m+\mu\sigma=0$ solution indeed generalizes to the full nonlinear theory.

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