Presence of Quantum Diffusion in Two Dimensions: Universal Resistance at the Superconductor-Insulator Transition

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We argue that whenever the transition between the insulating and superconducting phases of a disordered two-dimensional Fermi system at zero temperature ($T=0$) is continuous, the system behaves like a normal metal right at the transition; i.e., the resistance has a finite, nonzero value at $T=0$. This value is *universal*—independent of all microscopic details. These features, consistent with recent measurements on disordered films, are hypothesized to apply to other 2D transitions at $T=0$, such as Anderson localization with spin-orbit coupling, and the quantum Hall effect.

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Since the realization that in two dimensions (2D) even small disorder localizes all electron states,[1] it has been understood that at zero temperature ($T=0$) electrons do not diffuse and a metallic phase with nonzero conductivity is forbidden in 2D. In this paper, however, we demonstrate from simple scaling arguments that right at a continuous $T=0$ phase transition separating superconducting from insulating behavior, disordered fermion films do have a finite, nonzero "metallic" resistivity (*at* $T=0$). We argue, moreover, that the value of the resistance (per square) right at the transition is *universal*, depending only on the universality class of the transition, and being insensitive to all microscopic details.[2] We believe that these results apply to recent experiments[3] on amorphous thin-film superconductors whose thickness is varied to tune through the superconductor-insulator transition. The experiments suggest that at a critical film thickness, the resistance approaches a nonzero constant as $T\rightarrow 0$. We also hypothesize that universal metallic resistance is a generic feature of continuous $T=0$ conducting-to-insulating transitions in 2D. For example, we expect universal values for the resistance at the 2D Anderson-localization transition[4] in the presence of strong spin-orbit scattering, and for the conductivities σ_{xx} and σ_{xy} at the transition between plateaus in the integer and fractional quantum Hall effects.[5] (The universal values characterizing these various transitions presumably differ from one another.)

Our description of the $T=0$ superconductor-insulator phase transition in thin amorphous[6] films such as those of Ref. 3 starts from the hypothesis that the transition is correctly described by a model of charge-$2e$ bosons moving in a 2D random potential. In the superconducting phase the electrons have bound to form Cooper pairs and a description of the low-energy physics in terms of charge-$2e$ composite bosons is presumably valid. In the insulating phase, where pairing is destroyed and the individual electrons are presumably localized by the disorder, such a description is clearly inadequate. It seems nonetheless likely that the asymptotic critical properties of the transition are insensitive to the obvious difference between bosonic and fermionic insulating phases, i.e., between the Bose-glass[7] and Fermi-glass[1] phases. Bose condensation and the superconducting transition in pure systems at finite T belong, e.g., in the same universality class, the difference between the normal bosonic and fermionic phases notwithstanding. Moreover, in 1D the $T=0$ superconductor-insulator transition can be studied directly in terms of a model of electrons with a BCS attractive interaction moving in a random potential. It is found[8] that the critical behavior of this $T=0$ phase transition is in the same universality class as the superfluid-insulator transition in a model of repulsively interacting bosons, representing the Cooper pairs, moving in a random potential. We therefore expect that the experimentally relevant superconductor-insulator transition in amorphous films can be properly described in terms of charge-$2e$ bosons.

Consider then the following imaginary-time action for such a system of bosons: $S=S_0+S_1$ with

$$S_0=\int d^dx\,d\tau[\psi^*\partial_\tau\psi+(\hbar/2m)|\nabla\psi|^2+U(x)|\psi(x,\tau)|^2], \tag{1a}$$

$$S_1=\int d^dx\,d^dx'\,d\tau V(x-x')[|\psi(x,\tau)|^2-n_0]\times[|\psi(x',\tau)|^2-n_0], \tag{1b}$$

Here $V(x)=(2e)^2/|x|$ is a repulsive Coulomb interaction between the bosons,[9] with n_0 a compensating positive-charge background (charge neutrality fixing the boson density at n_0), and $U(x)$ a random potential.

For a given disorder strength, as the boson density n_0 is increased through some critical density n_c, we expect a $T=0$ phase transition from a localized Bose-glass phase[7] to a superconducting phase with $\langle\psi\rangle\neq 0$. Provided this transition is continuous, it is characterized by a superconducting correlation length which diverges as $\xi \sim \delta^{-\nu}$, where $\delta = n_0 - n_c$ measures the distance to the transition and ν, the correlation-length exponent, satisfies the inequality[7] $\nu \geq 2/d$. There is also a characteristic frequency Ω which vanishes at criticality as $\Omega \sim \xi^{-z}$, where z is the dynamical exponent.[7] Near the $T=0$ critical point all frequencies and the temperature scale[7] with Ω. Thus the *finite*-temperature superconductor-to-normal transition occurs at a temperature T_c, which scales as[7] $T_c \sim (n_0 - n_c)^{z\nu}$ for $n_0 \to n_c^+$.

In the superfluid phase the second-sound (phonon) mode has a plasmon-like dispersion relation[10,11] $\omega \sim k^{(3-d)/2}$ due to the long-range Coulomb interaction between the bosons. This mode can be described by an effective imaginary-time action[7] which depends only on the phase ϕ of the order parameter $\psi = |\psi| \exp(i\phi)$,

$$S_\phi = \frac{1}{2} \int d^dk\, d\omega [(\rho_s \hbar/2m) k^2 + \hbar\omega^2 |k|^{d-1}/e_R^2] |\phi(k,\omega)|^2 . \qquad (2)$$

Here ρ_s is the fully renormalized superfluid density and e_R a "fully renormalized" charge,

$$e_R^2 \equiv \lim_{k\to 0} |k|^{d-1}/C_{nn}(k,\omega=0) ,$$

with $C_{nn}(k,\omega) = \delta\langle n(k,\omega)\rangle/\delta\mu(k,\omega)$ the density-density response function. Near the $T=0$ superconductor-insulator transition, ρ_s vanishes as $\rho_s \sim \xi^{-(d+z-2)}$. This follows from the fact[7] that the contribution to the action from a coherence volume ξ^d/Ω of the first term in (2) is of order $\hbar$. Similar reasoning applied to the second term implies that the charge e_R should scale near the transition as $e_R^2 \sim \xi^{1-z}$. However, in a charged system, e_R cannot be zero, even in an insulating phase, implying the bound $z \leq 1$ on z. Provided the insulating phase is a gapless Bose glass, rather than a Mott-Hubbard insulator with a gap[7] (in which case one can have $e_R = \infty$), we expect e_R will approach a finite value at the transition so that $z=1$. The result $z=1$, which should hold in all dimensions, is the generalization to charged systems of the relation $z=d$, which has been argued to hold[7] at the $T=0$ superfluid-insulator transition in charge *neutral* boson systems, such as ^{4}He in porous media.

Scaling of the frequency-dependent conductivity near the superconductor-insulator transition can be obtained from the relation[12]

$$\sigma(\omega) = (2e)^2 \rho_s(-i\omega)/(-im\omega) , \qquad (3)$$

where $\rho_s(\omega)$ is a generalized frequency-dependent superfluid density defined as

$$\rho_s(\omega) = \langle |\psi|^2 \rangle = (m/4e^2\hbar) \int d^dx\, d\tau \langle T_\tau J_x(x,\tau) J_x(x',\tau')\rangle \exp[i\omega(\tau - \tau')] , \qquad (4)$$

with J_x the x component of the charge-$2c$ boson current operator, $J = -(e\hbar/mi)[\psi^* \nabla\psi - \psi\nabla\psi^*]$. Since all frequencies should be scaled by the characteristic frequency Ω near the transition one can write the scaling relation

$$\rho_s(\omega,\xi) = \xi^{-d} (\xi/a)^{2-z} \tilde{\rho}_s(\omega/\Omega) , \qquad (5)$$

where $\tilde{\rho}_s$ is an appropriate dimensionless scaling function and $\Omega = (\hbar/ma^2)(a/\xi)^z$, with a a short-distance cutoff. For $x \equiv \omega/\Omega \to 0$ we must recover the result $\rho_s \sim \xi^{-(d+z-2)}$, so $\tilde{\rho}_s(x)$ must approach a constant. The form as $x \to \infty$ is set by the requirement that at criticality, where both ξ and $\Omega^{-1} \sim \xi^z$ are infinite, $\rho_s(\omega, \xi=\infty)$ is finite: $\tilde{\rho}_s(x) = c_d x^{(d+z-2)/z}$, with c_d a dimensionless constant. Combining this with (3) we deduce that at criticality

$$\sigma(\omega,\xi=\infty) = c_d (e^2/\hbar) a^{2-d} (-i\hbar\omega/ma^2)^{(d-2)/z} . \qquad (6)$$

Similarly, at criticality, the finite-temperature dc conductivity should scale as $\sigma(T,\xi=\infty) \sim T^{(d-2)/z}$. This is consistent with the result $\sigma \sim 1/T$, derived by Giamarchi and Schulz[8] for the 1D superconductor-insulator transition (with disorder but short-range interactions), for which $z=1$.

In 2D the $T=0$ conductivity at criticality is therefore a finite constant, $c_d e^2/\hbar$, in the dc limit. Thus, at the superconductor-insulator transiton the system exhibits true *metallic* conduction at $T=0$, something *not* possible in 2D normal fermion systems. The Cooper pairs, poised on the brink of becoming superconducting, are capable of ordinary diffusion. Note that scaling, which gives $\sigma(\omega) \sim \omega^{(d-2)/d}$ for the Anderson-localization transition for electrons, appears to imply a finite dc σ at criticality for that case as well. This is illusory, however: Since $d=2$ is the lower critical dimension for the localization transition for fermions without spin-orbit coupling, logarithmic corrections in ω drive the conductivity to zero, localizing all the states.[1] Logarithms are *not* expected in the boson case (6), however, since $d=1$, not $d=2$, is the lower critical dimension.[7,11]

Recall the resistance per square, $R^* \equiv 1/\sigma(\omega=0,\xi=\infty)$, when expressed in units of h/e^2 is a pure number [$1/2\pi c_d$ in (6)], given by the $k=\omega=0$ limit of the response function defined in (3) and (4), evaluated *at* the critical point. Standard renormalization-group (RG) arguments imply that this number is, like critical exponents, *universal*, depending only on the universality class of the relevant transition, and not on microscopic details. The situation is rather analogous to what happens at the Kosterlitz-Thouless transition in the 2D classical *XY* model, where the dimensionless ratio $\hbar^2\rho_s(T_c)/mk_BT_c$ is a universal number[13] [cf. Eq. (3) with T_c replaced by $\hbar\omega$].

Similar arguments presumably apply to other $T=0$ conducting-to-insulating transitions, leading us to hypothesize that, e.g., the conductivity at the Anderson-localization transition in the presence of spin-orbit coupling[4] in 2D and σ_{xx} and σ_{xy} at the transition between plateaus in the quantum Hall effect[5] are also universal.

Will the universal resistance at the superconductor-insulator transition be a simple rational times the quantum of resistance $h/4e^2$? As detailed below, bosons and vortices (in the boson wave function) play a dual role at the transition. In the superconducting phase, the vortices are bound into pairs and do not conduct, whereas in the insulating phase the vortices are mobile and (Bose) condense. Since each boson carries charge $2e$, and each 2π phase slip induced by vortex motion causes a time-integrated voltage drop in units of $(\hbar/2e)2\pi = h/2e$, if the transiton was *self*-dual between bosons and vortices, R^* would be exactly $h/4e^2$. As argued below, however, the theory is probably *not* self-dual, so a simple rational times $h/4e^2$ seems unlikely.

We now sketch the derivation of a field theory, and the RG analysis, near the upper critical dimension of 6, of the fixed point governing the superconductor-to-Bose-glass transition at $T=0$ for model (1). We start from a lattice model since it provides a slightly more convenient representation[7,14] than does the continuum model (1):

$$H \equiv \sum_{ij} V_{ij}(\hat{n}_i - n_0)(\hat{n}_j - n_0) + \sum_i U_i \hat{n}_i - t\sum_{i,v}\cos(\Delta_v \hat{\phi}_i) . \tag{7}$$

Here $\hat{n}_i$, the boson number on site i, is conjugate to the order-parameter phase: $[\hat{\phi}_i, \hat{n}_j] = i\delta_{ij}$; t is the hopping strength, U_i and $V_{ij} \sim |i-j|^{-1}$ are, respectively, the lattice equivalents of the random and Coulomb potential in (1), and $\Delta_v \equiv \hat{\phi}_{i+v} - \hat{\phi}_i$.

Representing the partition function Z associated with (7) as a path integral over a basis of states diagonal in the density, $\hat{n}$, and repeating the duality transformations described in Ref. 14 for $d=2$ [the long-range Coulomb interactions in (7) being the only new feature], one can write Z in terms of a Hamiltonian describing an effective "superconductor" coupled to a gauge field, $\mathbf{A}$, which is simply related[14] to the original boson density. Because of the duality transformations, the *disordered* phase of this system corresponds to the *superfluid* phases of the original boson problem, while ordered, "superconducting" phases correspond to insulating phases of the boson system. Reference 14 describes these correspondences, and shows that the "superconducting" order parameter of the dual theory represents the vortices in the original boson wave function. The Bose-glass phase is characterized by a vortex order parameter which is nonzero but varies from site to site, vanishing under spatial averaging. The complex, Hermitian matrix $Q_{\alpha\neq\beta}(\mathbf{x},\tau,\tau')$, an analog of the Edwards-Anderson order parameter for spin glasses,[15] is thus a more appropriate order parameter. (The replica subscripts α and β run from 1 to n, the familiar limit $n\rightarrow 0$ to be taken at the end of the calculations.[15]) The static character of the quenched impurities in (7) is responsible for the nonlocality in time[7] of $Q_{\alpha\neq\beta}$, which distinguishes this order parameter from the closely analogous objects that characterize so-called "gauge glasses."[16] Writing Z in terms of $Q_{\alpha\neq\beta}$ and the replicated version of the gauge field $\mathbf{A} \equiv (\mathbf{A}^\perp, A^\tau)$ yields the final action $S' = S'_0 + S'_1$:

$$S'_0 = \sum_\alpha \int d^dx\, d\tau \left(\frac{1}{2t} |(\nabla\times\mathbf{A}_\alpha)^\perp|^2 + \int d^dx'\, V(x-x')[(\nabla\times\mathbf{A}_\alpha)^\tau_x - n_0][(\nabla\times\mathbf{A}_\alpha)^\tau_{x'} - n_0] \right) , \tag{8a}$$

$$S'_1 = \tfrac{1}{2} \sum_{\alpha\neq\beta} \int d^dx\, d\tau\, d\tau' [\,|\mathbf{D}^\perp_{\alpha\beta} Q_{\alpha\beta}(x,\tau,\tau')|^2 + |D^\tau_\alpha Q_{\alpha\beta}|^2 |D^{\tau'}_\beta Q^*_{\alpha\beta}|^2 + r|Q_{\alpha\beta}|^2)] + w\int d^dx\, \mathrm{Tr}[Q(x)^3] . \tag{8b}$$

Here $\mathbf{D}^\perp_{\alpha\beta} \equiv \nabla_\perp - ie'[\mathbf{A}^\perp_\alpha(\tau) - \mathbf{A}^\perp_\beta(\tau')]$, $D^\tau_\alpha \equiv \partial_\tau - ie' \times A^\tau_\alpha(\tau)$, $\nabla \equiv (\nabla_\perp, \partial_\tau)$, and e', r, and w are coupling constants. The trace in (8b) is over replica indices and time. The obvious dissimilarity of representations (8) and (1) make it highly unlikely that the fixed points of these two actions are identical, and hence that the superconductor–Bose-glass transition is self-dual. Thus we do not expect R^* to be a rational multiple of $h/4e^2$.

Keeping in mind that only for $d=2$ is (8) dual to the original problem (1), one can analytically continue the dimensionality d to perform a momentum-shell RG analysis. Since the disorder is correlated in time, such analysis requires a double expansion in the two small parameters[7] $\varepsilon \equiv 6-d$ (6 being the upper critical dimension), and ε_τ, the number of time dimensions. The physical problem of interest corresponds to $\varepsilon=4$ and $\varepsilon_t=1$. To leading order one obtains the correlation length, spatial decay, and dynamical exponents, $\nu = \frac{1}{2} + 5\varepsilon/24$, $\eta = \varepsilon_\tau = \varepsilon/6$, and $z = 1-\varepsilon/6$, respectively. Unfortunately, this dimensionality expansion is not helpful in estimating the "universal conductivity," since the conductivity is only universal *in* $d=2$.

In the absence of a direct calculation of R^* at the superconductor–Bose-glass transition in 2D, it is instructive to calculate the corresponding universal quantity for related models. For example, for integer n_0, the lattice model (7) with no disorder ($U_i=0$) has an insulating phase which is a Mott insulator with a particle-hole gap, rather than a Bose glass.[7] The Mott-insulator–superconductor transition at fixed integer density is governed,[11] for $D=2$, by the isotropic 3D XY fixed point. Since even weak disorder presumably[7] produces a gapless Bose-glass phase between the Mott and superconducting states, the 3D XY model does not describe the superconducting transition in any real disordered material. Nevertheless, in systems with integer density and weak disorder (e.g., periodic arrays of Josephson junctions), the model might well give a reasonable description of the transition, except asymptotically close to criticality.

The universal resistance at the *Mott*-insulator–to–superconductor transition follows by evaluating (3) and (4) at the critical point of the classical XY model [i.e., Eq. (1) with $U(\mathbf{x})=0$, $V(\mathbf{x})=\delta(\mathbf{x})$, and $\partial_\tau \rightarrow -\partial_\tau^2$]. To get a feeling for the size of R^*, we have generalized the O(2) XY model to O($2N$), by considering a model with N complex fields, and have calculated the resistance exactly in the $N=\infty$ limit; in this limit the Hartree approximation becomes exact. The result at criticality for R^* is $R^*=(8/\pi)R_Q$, where $R_Q=h/4e^2=6.5$ kΩ/□ is the quantum unit of resistance. This exactly solvable limit confirms the scaling analysis leading to (6) and hence supports the claim that amorphous films have a universal resistance at the superconductor-insulator transition.

Although one should not use the XY-model numbers seriously in comparison with experiment, it is amusing that they are consistent with current experimental data. Recent experiments on Josephson-junction arrays,[17] the system for which the Hamiltonian (8) with integer n_0 and no randomness seems most appropriate, appear consistent with $8R_Q/\pi$. Data on this amorphous films[3] suggest R^*'s in the range 8–25 kΩ, again not inconsistent with $8R_Q/\pi$. Of course the experiment must be taken to lower temperatures before experimental confirmation of the universality of R^* can be claimed, and the universal value measured. Likewise, a theoretical advance is necessary to calculate a value of R^* more reliable than "something of order R_Q."

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