Wave Propagation through Disordered Media and Universal Conductance Fluctuations

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Using the many-channel approximation to Landauer's formula and statistical scattering theory, we calculate analytically the average, $\langle G\rangle$, and the variance, var(G), of the conductance of a disordered sample of length L, described by a microscopic random Hamiltonian, and coupled at each end to an ideally conducting lead. We show that the coupling to the leads strongly affects the behavior of $\langle G\rangle$ and var(G) for sample sizes $L \lesssim L_0$, where L_0 is a characteristic length of the order of several tens of the elastic mean free path. For $L \gg L_0$, this coupling becomes unimportant.

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Measured at low temperature, as a function of an external magnetic field or of the Fermi energy, E_F, the conductance, G, of a disordered wire of mesoscopic size displays universal fluctuations in the metallic regime:[1] The amplitude of the fluctuations is of order one in units of e^2/h irrespective of the mean value $\langle G\rangle$. This phenomenon has prompted considerable theoretical activity, and has been studied from various points of view, including diagrammatic perturbation theory,[2] numerical simulation,[2,3] and the statistics of the transfer matrix.[3,4] Because of the approximations used, the above-mentioned analytical methods are essentially all confined to sample lengths, L, much larger than the elastic mean free path, l. Using the many-channel approximation to Landauer's formula and a statistical scattering theory recently developed in the context of nuclear physics, we present in this Letter a novel approach to the problem which enables us to calculate $\langle G\rangle$ and the variance, var(G), analytically and nonperturbatively (with respect to the impurity averaging) for all length scales L between a few times l and the localization length.

We consider a disordered sample coupled to two ideally conducting leads, at zero temperature, modeling the disorder microscopically in terms of a random Hamiltonian. Most importantly, we take realistic account of the coupling between the disordered sample and the two leads—necessary to define and measure the conductance. Our results for $\langle G\rangle$ and var(G) show that this coupling, treated only cursorily in most previous work, is essential for a complete understanding of universal conductance fluctuations. Although irrelevant for sufficiently large samples ($L \gg l$), it constitutes an important factor in the behavior of both $\langle G\rangle$ and var(G) for values of L up to several tens of l. It introduces into the problem (which is commonly discussed entirely in terms of the Thouless energy, E_c) a new energy scale—the width, Γ, for electron emission from the sample into the leads. Any open system is characterized[5] by such a decay width Γ.

The system consists of a disordered sample of length L, connected at either end to infinite perfectly conducting leads, all taken to lie along the x axis. Electrons propagate along the leads in the $\pm x$ direction, populating one of a finite number, $\Lambda \simeq L_y L_z k_F^2/\pi$, of transverse modes, where L_y, L_z are the transverse widths of the lead[6] and k_F the Fermi wave number. Typically, $\Lambda \gg 1$. We divide the disordered region into $K = L/l$ equal slices. Physically, the slices characterize successive elastic scatterings of the electron as it moves through the sample. The thickness of each slice in the x direction is accordingly chosen to be l, rendering scattering events in different slices statistically independent. We consider samples of length sufficient to exclude ballistic electrons, and confine ourselves to the domain of weak localization where $k_F l \gg 1$. With each slice, we associate a site at which there are N possible states generated by a random Hamiltonian matrix of dimension N. Qualitatively, we have $N \simeq \Lambda k_F l \gg \Lambda$. Since we consider neither external magnetic fields nor spin-orbit scattering, the system is time-reversal invariant. Therefore, we assume the random Hamiltonian matrix to be a member of the Gaussian orthogonal ensemble (GOE). Matrices at different sites are taken to be uncorrelated. Propagation of the electron through the disordered sample is then provided by nearest-neighbor hopping between sites, while propagation from either lead into the sample, and vice versa, is furnished by dynamically coupling the left (right) lead to the first (last) site, respectively. To define the model completely, boundry conditions must be chosen in the two leads at the sample end points. Fortunately, the conductance turns out to be independent of this choice.

It is convenient to define the Hamiltonian, H, of the total system as the sum of two pieces, $H = H_0 + V$. Here, V comprises the site-site and the site-lead hopping and, for the purposes of scattering theory, may be regarded adiabatically switched on (off) for time $t \to -\infty$ ($t \to +\infty$), respectively. The Hamiltonian H_0 thus describes a totally disconnected system of two semi-infinite leads and K sites. In the absence of transmission ($V = 0$), flux conservation demands that the allowed eigenstates, $|\chi^c_{E,a}\rangle$, in the leads correspond to waves totally reflected at the end points, a_c, with $c = L, R$. We suppress the elastic phase shift since it does not contribute to G. The channel states, $|\chi^{L(R)}_{E,a}\rangle$, have zero amplitude in the right (left) lead; where $a = 1, 2, \ldots, \Lambda$ is the

channel index for the transverse modes, while the continuous energy, E, is given by $H_0|\chi^c_{E,a}\rangle = E|\chi^c_{E,a}\rangle$. At each site, i, for $i=1,2,\ldots,K$, there exists a basis of N orthornormal states, $|i\mu\rangle$, with $\mu=1,2,\ldots,N$. For $\mu \geq \nu$, the matrix elements $\langle i\mu|H_0|j\nu\rangle = \delta_{ij}H^{(ii)}_{\mu\nu}$ are uncorrelated random variables with a Gaussian probability distribution centered at zero, and having a second moment $\langle (H^{(ii)}_{\mu\nu})^2\rangle = (1+\delta_{\mu\nu})\lambda^2/N$. The angular brackets denote the ensemble average, and the parameter λ defines (for $V=0$) the mean level spacing (which is the same for all sites).

We assume that the real matrix elements of V connecting two neighboring sites, $H^{(i,i+1)}_{\mu\nu} = \langle i\mu|V|i+1\nu\rangle$, are real uncorrelated Gaussian-distributed random variables centered at zero, with a second moment given by $\langle (H^{(i,i+1)}_{\mu\nu})^2\rangle = v^2/N$. Hopping elements connecting different pairs of sites are uncorrelated, and so are $H^{(ii)}_{\mu\nu}$ and $H^{(j,j+1)}_{\mu'\nu'}$. Setting $H^{(ij)}_{\mu\nu}=0$ for $|i-j|>1$, we have now fully defined the matrix ensemble $H^{(ij)}_{\mu\nu}$, which we observe to be equivalent to Wegner's gauge-invariant disorder.[7] We have also worked out G for a model with site-diagonal disorder and constant hopping matrix elements, $H^{(i,i+1)}_{\mu\nu}$, and obtained identical results for sufficiently weak hopping. We complete the model by noting that the matrix elements of V between any pair of channel states must vanish, and by assuming the real matrix elements

$$\langle i\mu|V|\chi^c_{E,a}\rangle = W^c_{a,\mu}(\delta_{i1}\delta_{cL}+\delta_{iK}\delta_{cR})$$

to be independent of E. This condition, required to hold approximately over an interval of typical length E_c, is fulfilled in most channels, since for $E_F \gg E_c$, the wave number changes little in such an interval. Furthermore, E_F should lie within the GOE spectrum and reasonably close to its center, which we have chosen to be zero.

Our model is formally very similar to the strong-coupling problem in the statistical theory of precompound nuclear reactions. Therefore, we defer the details of our calculation to a future publication, and simply focus attention on the novel aspects. The interested reader can find the steps necessary for the present calculation in Ref. 8.

To calculate the dimensionless conductance, $g=(h/e^2)G$, we use linear-response theory[9] and obtain, for $\Lambda \gg 1$, $g=\sum_{a,b}[|S^{LR}_{ab}|^2+|S^{RL}_{ab}|^2]$, i.e., the many-channel Landauer formula. A factor of 2 accounts for the spin degeneracy of the electron. The symbol S^{LR}_{ab} denotes the element of the S matrix, taken at the energy E_F, for a particle coming in on the right in channel b and going out on the left in channel a. Time-reversal invariance implies that $S^{RL}_{ab}=S^{LR}_{ba}$.

In our model, the kernel of the Lippmann-Schwinger equation is of finite rank. Thus $S^{cc'}_{ab}$ can be calculated algebraically,[10] and we obtain the manifestly unitary form

$$S^{cc'}_{ab} = \delta^{cc'}\delta_{ab} - 2i\pi\sum_{\mu,\nu} W^c_{a,\mu}(D^{-1})^{cc'}_{\mu\nu}W^{c'}_{b,\nu}, \quad (1)$$

where

$$D^{ij}_{\mu\nu} = E^+\delta_{ij}\delta_{\mu\nu} - H^{(i,j)}_{\mu\nu} + i\delta_{ij}[\delta_{i1}\Omega_{\mu\nu}(L)+\delta_{iK}\Omega_{\mu\nu}(R)], \quad (2)$$

and the superscripts (c,c') on D^{-1} in Eq. (1) should be read as $L\rightarrow 1$, $R\rightarrow K$. The matrices $\Omega_{\mu\nu}(c)$, $c=L,R$, given by $\pi\sum_a W^c_{a,\mu}W^c_{a,\nu}$, describe the coupling to the open channels and cause the eigenvalues, $\mathcal{E}_a$, of D (i.e., the poles of $S^{cc'}_{ab}$) to be *complex*. The width Γ, defined below, is essentially the average of $-2\,\mathrm{Im}(\mathcal{E}_a)$. Inserting Eq. (1) into the Landauer formula, we find that g can be written as a sum of two traces containing products of matrices of dimension N. This fact makes it possible to express g (g^2) as the second (fourth) derivative, with respect to suitable auxiliary variables, of a graded determinant and, hence, of a generating function, Z, involving integration over both commuting and anticommuting variables.[11,12] After ensemble averaging Z, we introduce composite variables and determine the saddle point. This procedure imposes a positive-definiteness requirement which entails $2v^2<\lambda^2$. For $\Lambda\gg 1$, the saddle point is unique since the coupling to the channels breaks the graded symmetry. Upon expanding the exponent in the integrand around the saddle point to second order in the generators of the composite variables, we carry out the resulting Gaussian integrals, dropping terms of order N^{-1} or higher. [This is formally equivalent to neglecting terms of order $(k_F l)^{-1}$, or higher, in the impurity diagram technique of Ref. 2.] In this way, we obtain $\langle g\rangle = 2T^L\Pi_{1K}T^R + \cdots$, where the dots indicate the weak-localization corrections to $\langle g\rangle$ given below, and a more complex expression for $\mathrm{var}(g)$ also involving T^L, T^R, and elements of the matrix Π. Before giving the details, we define and interpret the quantities T^L, T^R, and Π_{ij}.

(i) Sticking probabilities.—The effective coupling between site 1 and the left lead (or site K and the right lead) is described by $T^c=\sum_a T^c_a$, where the "sticking probabilities," T^c_a, are given[13] by $1-|\langle S^{cc}_{aa}\rangle|^2$. The T^c_a measure the unitarity deficit of the average S matrix and obey $0\leq T^c_a\leq 1$. Identifying the ensemble average with the running average over energy for a single member of the ensemble (ergodic hypothesis), we observe that T^c_a measures that part of the incident flux in channel a which is not instantaneously reemitted back into the incident channel. Only this part has the opportunity to diffuse through the disordered sample, and thereby to contribute to the conductance. All the matrix elements, $W^c_{a,\mu}$, appearing in Eqs. (1) and (2) have been absorbed into the T^c_a. The width Γ for electron emission back into the leads is given by $(T^L+T^R)/2\pi\rho K$. Here, $\rho \simeq N/\pi\lambda$ is the average level density at the Fermi energy at site j as obtained from the saddle-point condition, which, for $v^2\ll\lambda^2$, is the same at each site; whence ρK is the total level density of the disordered sample at the Fermi energy. We focus our attention on the regime of "strong ab-

sorption," $\langle S_{aa}^{cc}\rangle \simeq 0$, so that[14] $T_a^c \simeq 1$ and $T^c \simeq \Lambda \gg 1$, whence $\Gamma\rho K \gg 1$ ("strongly overlapping levels").

(ii) Diffusion propagator.—The quantity Π_{1K} is the $(1,K)$ element of a matrix, Π, which is intimately related to the diffusion propagator studied in Ref. 2. It is defined in terms of the dimensionless coupling between neighboring sites, $x=(2\pi\rho)^2 v^2/N$. The couplings T^L and x, between site 1 and the left lead and site 2, respectively, are equally strong when $x \simeq \Lambda$, whence $v^2/\lambda^2 \simeq \Lambda/N \simeq (k_F l)^{-1} \ll 1$. In this case, we can use first-order perturbation theory to account for v^2. To this order of approximation, we have

$$(\Pi^{-1})_{ij} = x[2\delta_{ij} - \delta_{i,j+1} - \delta_{j,i+1}] + \delta_{ij}[\delta_{i1}(T^L - x) + \delta_{iK}(T^R - x)]. \quad (3)$$

Putting $T^L \simeq T^R \simeq \Lambda$, and with $D = 2\pi\rho v^2 l^2/\hbar N$, we then find from Eq. (3)

$$\Pi_{1K} = \{2\Lambda[1 + \tfrac{1}{4}K(K-1)l^2\Gamma/\hbar D]\}^{-1}. \quad (4)$$

To establish the connection with the diffusion propagator, let us first consider a "closed system," defined by $T^c = 0$. In this case, the eigenvalues of Π^{-1} are given by $2x\{1 - \cos[\pi(n-1)/K]\}$, $n = 1,2,\ldots,K$. For $K \gg 1$ and small n, and in units of $2\pi\rho$, they are approximated by $(n-1)^2 E_c$, where $E_c = \pi^2\hbar D/L^2$ is the Thouless energy. For $T^c \neq 0$ but $\Gamma \ll E_c$, we use perturbation theory to find that up to first order, the shifted eigenvalues are given by $(2 - \delta_{n,1})\Gamma + (n-1)^2 E_c$. We note that the lowest eigenvalue is $\Gamma \neq 0$, and changes continuously with the strength of the coupling to the leads. It can be shown that the time evolution of the electronic occupation probability (for an electron initially at site 1) obeys a master equation, with $(-\Pi^{-1})$ as the generating matrix. In the continuum limit ($K \to \infty$), this equation tends towards a diffusion equation with the above-defined D as the diffusion constant.

(iii) Results.—With $T_a^c \simeq 1$, we find

$$\langle g\rangle = \frac{2\Lambda}{2+\gamma(K-1)} - \frac{2}{3}\left[1 + \sum_{n=1}^{3} \frac{\alpha_n}{[2+\gamma(K-1)]^n}\right] + O(\Lambda^{-1}), \quad (5)$$

where $\alpha_1 = -3\gamma$, $\alpha_2 = 2\gamma(\gamma+3)$, $\alpha_3 = -2(2\gamma^2+1)$, and $\gamma = \frac{1}{2}Kl^2\Gamma/\hbar D$. The first term on the right-hand side of Eq. (5), equivalently expressed as $2T^L\Pi_{1K}T^R$, gives the leading contribution to $\langle g\rangle$, while the second term on the right-hand side is the weak-localization correction which, for $L \gg l$, attains the well-known value[2,4] of $\frac{2}{3}$. The limit of validity of the asymptotic expansion is reached when K is so large that both terms become about equal. This is the case for $K \simeq 3x \simeq 24\Lambda$, i.e., when L becomes comparable with the localization length, $L_{\rm loc} \simeq \Lambda l$. Our model does not allow for ballistic electrons. That is why for $K = 1$, $\langle g\rangle$ is only half the quantized contact conductance of 2Λ.

The term $2T^L\Pi_{1K}T^R$ consists of the probabilities, T^c, to enter or leave the disordered sample, and the probability, Π_{1K}, to diffuse through it. To elucidate the behavior of $\langle g\rangle$, we put $(K-1)/K \simeq 1$, and write the contents of the square brackets in Eq. (4) alternatively as $1 + \frac{1}{4}\pi^2\Gamma/E_c$ or as $1 + L/L_0$, where $L_0 = 4\hbar D/l K\Gamma \simeq L_{\rm loc}/\Lambda$. (Using $D = v_F l$, where v_F is the Fermi velocity, we obtain $L_0 \simeq 16l$.) The second form shows that for $L \lesssim L_0$, $\langle g\rangle \simeq \Lambda$ is not Ohmic and nearly independent of L. For $L \gg L_0$, it acquires the Ohmic structure $\langle g\rangle \simeq 4\pi\hbar\rho D/lL$, which is independent of the coupling to the leads. The first form displays the reason for the different modes of behavior: For $L \lesssim L_0$, we have $\Gamma \lesssim E_c$ and so the time, $\hbar/E_c$, for diffusion through the disordered sample is small compared with the emission time, $\hbar/\Gamma$, or similar to it. In a time-dependent picture, the electronic occupation probability, fed into the lead by the incident flux, has time to distribute itself uniformly across the sample before decaying back into the leads. Since reflection and transmission are equal, $\langle g\rangle$ almost attains the maximum value, Λ. For $L \gg L_0$, most of the incident flux is reemitted into the incident channel and only a small fraction diffuses through the sample, in which case $\langle g\rangle \ll \Lambda$.

For the variance, we find

$$\mathrm{var}(g) = \frac{8}{15}\left[1 + \sum_{n=3}^{6} \frac{\beta_n}{[2+\gamma(K-1)]^n}\right] + O(\Lambda^{-1}), \quad (6)$$

where $\beta_3 = 15\gamma^3$, $\beta_4 = -2\gamma^3(8\gamma+15)$, $\beta_5 = -4(2\gamma^4 + 10\gamma^2 + 3)$, and $\beta_6 = 20(4\gamma^4 + 4\gamma^2 + 1)$. We observe that, independent of γ, $\mathrm{var}(g) \to \frac{1}{2}$ for $K \to 1$, while as $K \to \infty$, $\mathrm{var}(g) \to \frac{8}{15}$. The dependence of $\mathrm{var}(g)$ on $K-1$, for $\gamma = \frac{1}{8}$ (which corresponds to taking $T_a^c = 1$ for all a,c), is shown as the solid line in Fig. 1. This line is remarkably smooth, and nearly constant. In construct-

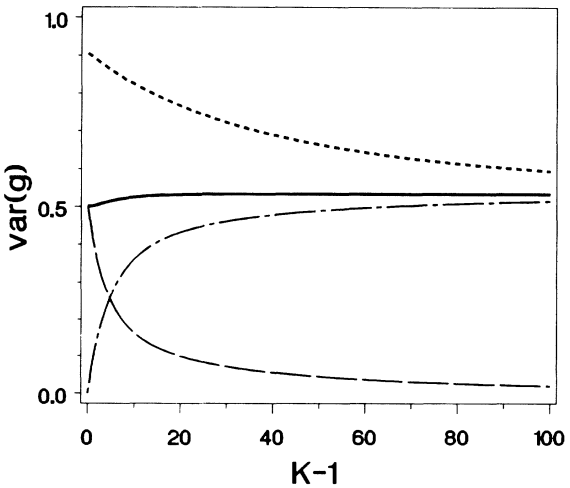

FIG. 1. The variance, $\mathrm{var}(g)$, as a function of the dimensionless length scale $K-1 = (L-l)/l$ as explained in the text.

ing the dashed and dash-dotted curves, we have decomposed var(g) into two contributions. These are defined as follows: The dashed line is the sum of all terms which have the form $(\Pi_{11})^l(\Pi_{1K})^m(\Pi_{KK})^n$, with l,m,n positive integers, whereas the dash-dotted line contains all contributions involving at least one summation over an intermediate site, i.e., a j summation over terms containing Π_{1j} and/or Π_{jK}. These definitions imply that the dashed line gives that part of var(g) which is solely due to the coupling with the channels. As L is increased, this part falls off with powers of E_c/Γ, indicating an ever-diminishing influence of this coupling. The remaining part is the contribution to the fluctuation arising from weak-localization effects, and attains the limiting value of $\frac{8}{15}$ for large K, consistent with previous analyses of var(g) for quasi-one-dimensional conductors.[2,4] We note that when $\Gamma \lesssim E_c$, the contribution to var(g) due to the coupling with the leads is significant in maintaining the universal character of the fluctuations.

Although the numerical value $L_0/l=16$ may be model dependent, and therefore uncertain, nonetheless L_0 characterizes a physically distinct length scale since, in general, L_0 is inversely proportional to T. Also, the fact that the channel coupling terms combine with the bulk terms in such a way as to yield a nearly flat curve for var(g) is specific to our choice $T_a^c=1$ for all a,c, and is not universal. This is demonstrated by the dotted curve in the figure, calculated for the choice $T_a^c=0.1$ for all a,c. We see that for small K, the fluctuations are enhanced by almost a factor of 2, and slowly fall off towards the asymptotic value of $\frac{8}{15}$. Such a situation may be realized experimentally by gating the disordered sample.

In summary, we have presented an approach to conductance fluctuations which takes proper account of the coupling to the channels, and we have displayed the influence of this coupling on both $\langle g\rangle$ and var(g). Moreover, we have shown that a random-matrix model for the Hamiltonian can successfully account for their universality when $L \gg l$. Although the presentation in this paper has been focused on the two-lead measurement, an extension to multilead devices and/or rings[1] is completely straightforward: The only modification required concerns the structure of the matrix Π^{-1} in Eq. (3), which must reflect the geometry of the sample and the presence of the leads. (The latter cause the occurrence of terms like T^c.)

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[13]We assume that $\langle S_{aa'}^{cc'}\rangle$ (which in any case is diagonal in c,c') is also diagonal in the channels a,a'. Lifting this assumption is possible only at the expense of complex notation and does not change the physical content of our results.

[14]One might argue that T^c can be increased arbitrarily by increasing the cross section of the leads without changing that of the sample. This, however, would only increase the number of channels poorly coupled to the sample, for which $T_a^c \simeq 0$, without changing T^c.