Aharonov-Bohm Scattering of Particles with Spin

C. R. Hagen
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 27 March 1989; revised manuscript received 30 October 1989)

The scattering of relativistic spin-one-half particles in an Aharonov-Bohm potential is considered. It is shown that earlier approaches to this problem have neglected a crucial delta-function contribution to the potential. By formulating the problem with a source of finite radius which is then allowed to go to zero, it is established that it is the delta function alone that causes solutions that are singular at the origin to become relevant. The changes in the amplitude which arise from the inclusion of spin are seen to modify the cross section for the case of polarized beams. Finally, the calculated Aharonov-Bohm amplitude is shown to describe the scattering of particles with arbitrary spin in the $c=\infty$ limit.

PACS numbers: 03.65.Bz

The Aharonov-Bohm effect[1] has long been recognized for its crucial role in demonstrating the importance of potentials in quantum mechanics. More recently, interest in this topic has been stimulated by the considerable effort currently being expended on the study of (2+1)-dimensional models in both superconductivity and particle theory. With the increased application of the results of Ref. 1 to other problems (e.g., Ref. 2) it has become important that certain limitations on the original Aharonov-Bohm calculation be removed. This has been accomplished in a recent series of papers.[3,4]

A concern which has been increasingly addressed of late has to do with the question of how the inclusion of spin modifies the results of Ref. 1. In particular, Alford and Wilczek[5] have applied their calculations for Dirac particles to the interaction of cosmic strings with matter. They assert that the scattering amplitude is unaffected by the spin, a result which certainly follows if one accepts their requirement that the upper component of the two-component spinor be regular at the origin. On the other hand, Gerbert and Jackiw[6] have suggested a more general boundary condition which introduces a new parameter into the calculation. This has been applied[7] to the question considered in Ref. 5 with quite different results being obtained. This paper approaches the same problem by attempting to infer the behavior of the wave function at the origin in terms of the underlying physics.

One begins by writing the Dirac equation for a particle of mass M which, in terms of two-component spinors ψ, is

$$E\psi=[M\beta+\beta\gamma\cdot\Pi]\psi\,, \tag{1}$$

where the matrices β and $\beta\gamma_i$ are conveniently defined in terms of the Pauli spin matrices as

$$\beta=\sigma_3,\quad \beta\gamma_i=(\sigma,s\sigma_2)\,,$$

and s is twice the spin value (+1 for spin "up" and -1 for spin "down"). The form (1) follows most simply by using the decoupling of the usual four-component Dirac equation in the absence of a third spatial coordinate into two uncoupled two-component equations for $s=+1$ and $s=-1$. This approach differs from the usual one which selects a particular value for s and will serve to make the results obtained more useful and transparent. The quantity Π_i is given by

$$\Pi_i=(1/i)\partial_i-eA_i\,,$$

where the potential A_i is related to the magnetic field in the usual way

$$H=\nabla\times A\,. \tag{2}$$

Since one is generally interested in the situation in which H is restricted to a flux tube of zero radius, it is conventional to write

$$eH=-(\alpha/r)\delta(r)\,, \tag{3}$$

with r being the two-dimensional radius vector. In the Coulomb gauge this yields for A_i the result

$$eA_i=\alpha\epsilon_{ij}x_j/r^2\,. \tag{4}$$

It is crucial to the solution of this problem that note be made of the fact that (1) and (2) imply

$$(E^2-M^2)\psi=-\gamma\Pi\gamma\Pi\psi=[\Pi^2+\alpha s\sigma_3(1/r)\delta(r)]\psi\,. \tag{5}$$

Equation (5) clearly shows that the individual components of ψ do not satisfy the equation solved in Ref. 1, but rather an equation with an additional delta-function interaction. This essential aspect of the problem has not been recognized in earlier attempts to include the element of spin.

In order to avoid complications arising from the fact that the delta function in (5) occurs at a singular point of the differential equation, one adopts here what is probably the only reasonable alternative, namely H is spread out over a region $r\le R$ such that

$$e\int Hr\,dr=-\alpha\,.$$

Although one is otherwise free to use any form of H [provided only that a contribution of the form (3) is excluded], the calculation is most easily performed if one replaces (3) by

$$eH = -(\alpha/R)\delta(r-R)$$

and (4) by

$$eA_i = \begin{cases} \alpha\epsilon_{ij}x_j/r^2, & r>R, \\ 0, & r<R, \end{cases}$$

with the understanding that the limit of $R \to 0$ is to be taken at the end of the calculation. Thus one considers (for a positive-energy solution) the set of equations

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right] f_m(r) = 0, \quad r<R,$$

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{(m+\alpha)^2}{r^2} + k^2\right] f_m(r) = 0, \quad r>R,$$

where $k^2 = E^2 - M^2$ and the $f_m(r)$ are the Fourier coefficients which occur in the expansion of the upper component ψ_1 of the total wave function, i.e.,

$$\psi_1 = \sum_{-\infty}^{\infty} a_m f_m(r) e^{im\phi}. \tag{6}$$

The effect of the delta function is then taken into account by means of the continuity relations

$$f_m(R-\epsilon) = f_m(R+\epsilon),$$
$$R\frac{d}{dr}f_m \Bigg|_{R-\epsilon}^{R+\epsilon} = \alpha s f_m(R). \tag{7}$$

The solution for $f_m(r)$ when $r<R$ is standard since normalizability (and/or the need to avoid the introduction of spurious delta functions at $r=0$) implies

$$f_m(r) = C_m J_{|m|}(kr), \quad r<R.$$

For $r>R$, $f_m(r)$ is required to have the form

$$f_m(r) = A_m J_{|m+\alpha|}(kr) + B_m J_{-|m+\alpha|}(kr),$$

where A_m, B_m, and C_m are constants and the J's are the usual Bessel functions. Upon applying (7) one obtains for the unnormalized $f_m(r)$ for $r>R$ (to lowest order in kR)

$$f_m(r) = (kR)^{|m|}\left[(kR)^{-|m+\alpha|}\left(\frac{1}{2} + \frac{|m|+\alpha s}{2|m+\alpha|}\right) J_{|m+\alpha|}(kr) + (kR)^{|m+\alpha|}\left(\frac{1}{2} - \frac{|m|+\alpha s}{2|m+\alpha|}\right) J_{-|m+\alpha|}(kr)\right]. \tag{8}$$

The $R \to 0$ limit then clearly implies that the $J_{-|m+\alpha|}(kr)$ term must drop out of (8) unless

$$|m+\alpha| = -|m| - \alpha s. \tag{9}$$

In that case one must include the next higher power of kR in the coefficient of the $J_{|m+\alpha|}(kr)$ term in (8). This yields

$$f_m(r) = (kR)^{|m|}\{D_m (kR)^{|m|+\alpha s+2} J_{|m+\alpha|}(kr) + (kR)^{-|m|-\alpha s} J_{-|m+\alpha|}(kr)\},$$

where D_m is a nonvanishing constant. One now infers that

$$f_m(r) \to J_{-|m+\alpha|}(kr)$$

in the limit $R \to 0$ provided that

$$|m| + \alpha s + 1 > 0 \tag{10}$$

or, upon using (9),

$$|m+\alpha| < 1.$$

This shows that the admissible solution of the Dirac equation in the limit $R=0$ is always the regular solution $J_{|m+\alpha|}(kr)$ unless (9) and (10) are simultaneously satisfied. Upon writing[4]

$$\alpha = N + \beta,$$

where N is an integer and

$$0 \le \beta < 1,$$

one can now establish that in Eq. (6) $f_m(r)$ is always $J_{|m+\alpha|}(kr)$ unless

$$m = -N, \quad N \ge 0, \quad s = -1 \tag{11a}$$

or

$$m = -N-1, \quad N+1 \le 0, \quad s = +1. \tag{11b}$$

As a check on the plausibility of the results obtained it may be noted that (9) implies that $\alpha s < 0$ in order that $J_{-|m+\alpha|}(kr)$ be an admissible solution. Since this is seen from Eq. (5) to be precisely the condition necessary for the delta function to be an attractive potential, one has the intuitively reasonable result that only an attractive delta function has the capability of making the solution more concentrated at the origin than is the regular solution $J_{|m+\alpha|}(kr)$.

Before going on to a calculation of the scattering amplitude some remarks are in order. In particular it is to be noted that in the nonrelativistic and in the relativistic spin-zero cases one must discard the αs term in (8) so that it is always the regular solution which is correct. Thus the Aharonov-Bohm amplitude is the standard result in these two cases. This simple observation allows one to conclude that there are no bound states in either of these applications since a bound state requires a delicate cancellation between the e^{-ikr} terms in the asymptotic expansions of the two solutions of Bessel's equation. The argument against bound states in the spin-one-half case is virtually identical and follows simply from the fact that it is always *either* $J_{|m+\alpha|}(rk)$ *or* $J_{-|m+\alpha|}(kr)$ which is correct.

One can now calculate the scattering amplitude in a fairly direct manner.[1,4] It is readily found that the phase shifts δ_m are identical to those of the spinless case, i.e., $\delta_m = -\frac{1}{2}\pi[|m+\alpha| - |m|]$ except for the cases (11) for which there is a sign reversal. The wave function ψ_1 is given by

$$\psi_1 = \sum{}' e^{-i(\pi/2)|m+\alpha|} J_{|m+\alpha|}(kr) e^{im\phi} + \theta(s)\theta(-\alpha) e^{-i(N+1)\phi} e^{-i(\pi/2)(\alpha - N - 1)} J_{\beta-1}(kr) + \theta(-s)\theta(\alpha) e^{-iN\phi} e^{-i(\pi/2)(N-\alpha)} J_{-\beta}(kr) , \quad (12)$$

where the prime on the summation indicates the omission of the two terms specified by Eq. (11) and generous use has been made of the step function

$$\theta(x) \equiv \tfrac{1}{2}(1 + x/|x|) .$$

An important observation to be made here is that when (1) is used to compute the lower component ψ_2 the integral of the scalar product $\psi^*\psi = \psi_1^*\psi_1 + \psi_2^*\psi_2$ over a region including the origin is found to exist even for the case when ψ_1 contains an irregular solution. It can also be seen that the Hamiltonian is self-adjoint. This is particularly satisfying since neither normalizability nor self-adjointness has been imposed on the solution.

Using (12) the amplitude can now be obtained as

$$f(\phi) = f_{AB}(\phi) + \left(\frac{2i}{\pi k}\right)^{1/2} \sin(\pi\alpha) e^{-iN(\phi-\pi)} \times e^{-i\phi/2} \theta(-\alpha s) e^{-i\phi\epsilon(s)/2} , \quad (13)$$

where $\epsilon(x) \equiv x/|x|$ and $f_{AB}(\phi)$ is the (corrected[4]) Aharonov-Bohm amplitude

$$f_{AB}(\phi) = -\left(\frac{i}{2\pi k}\right)^{1/2} \frac{\sin(\pi\alpha) e^{-iN(\phi-\pi)}}{\cos\phi/2} e^{-i\phi/2} .$$

(It has been assumed that the incident wave is from the right.) Worth noting is the invariance of (13) under $\alpha \to -\alpha$, $\phi \to -\phi$, and $s \to -s$, thereby verifying an important symmetry implied by Eq. (5). The result (13) can be written more compactly as

$$f(\phi) = -\left(\frac{i}{2\pi k}\right)^{1/2} \frac{\sin(\pi\alpha) e^{-iN(\phi-\pi)}}{\cos\phi/2} \times e^{-i\phi/2} \epsilon(\alpha s) e^{-i\phi\epsilon(s)\theta(-\alpha s)} , \quad (14)$$

while the complete two-component wave function is given by the asymptotic form

$$\psi = \begin{pmatrix} 1 \\ -k/(E+M) \end{pmatrix} e^{-ikr\cos\phi - i\alpha\phi} + \begin{pmatrix} 1 \\ k/(E+M) e^{is\phi} \end{pmatrix} r^{-1/2} e^{ikr} f(\phi) . \quad (15)$$

From Eq. (14) it follows that the scattering cross section, while identical to the standard Aharonov-Bohm result for unpolarized beams, is modified when the spin of the scattered particle is preferentially oriented in a direction perpendicular to the flux tube. Thus for a beam polarized along the direction of the unit vector $\mathbf{n}$ one finds that

$$\frac{d\sigma}{d\phi} = \frac{\sin^2(\pi\alpha)}{2\pi k \cos^2\phi/2} [1 - (\mathbf{n}\times\hat{\mathbf{z}})^2 \cos^2\phi/2] ,$$

where $\hat{\mathbf{z}}$ is a unit vector parallel to the flux tube. (Note that in this expression $\mathbf{n}$ and $\hat{\mathbf{z}}$ are three-dimensional vectors.) In the optimal case in which $\mathbf{n}$ and $\hat{\mathbf{z}}$ are perpendicular there occurs a complete cancellation in the cross section for backward scattering (i.e., $\phi = 0$). For other choices of $\mathbf{n}$ the spin effect is an energy-dependent but angle-independent reduction of the scattering cross section. Measurement of such effects may well be experimentally feasible.

The existence of singular states as displayed in Eqs. (11) and (12) has another important consequence which deserves mention here. It has already been pointed out that bound states cannot occur in this system even with the inclusion of spin. The remarkable thing, however, is that the spin effect puts one precisely at the threshold of binding in channels described by the quantum numbers (11). In the usual case the condition that the wave function ψ vanish at the origin requires that the potential must exceed a certain minimum strength in order for ψ to be capable of being smoothly joined to a decaying exponential as is necessary for the existence of a bound state. In one spatial dimension, of course, there is no requirement that ψ vanish at the origin and consequently arbitrarily weak attractive potentials lead to binding. In the present application the occurrence of states which have singular wave functions at the origin means that arbitrarily weak attractive potentials must lead to bound states. This is an aspect which has an obvious and potentially far reaching impact on superconductivity applications.

A significiant extension of the domain of applicability of Eq. (14) is accomplished by noting that the Galilean limit of (1) is obtained by setting $E = M + \mathcal{E}$ and letting $2M + \mathcal{E} \to 2M$. The resulting equation

$$[\mathcal{E}\tfrac{1}{2}(1+\sigma_3) + M(1-\sigma_3) - \sigma_1\Pi_1 - s\sigma_2\Pi_2]\psi = 0 \quad (16)$$

is covariant with respect to Galilean transformations in 2+1 space and is, in fact, the Galilean spin-one-half equation of Lévy-Leblond[8] when the third spatial coordinate is absent. One thus can infer immediately that Galilean spin-one-half particles are described by Eqs. (14) and (15) provided that $E+M$ is replaced by $2M$ and the wave number k takes its "nonrelativistic" value $(2M\mathcal{E})^{1/2}$. While this is of no particular interest in itself (since a limit of a more general expression is involved), the significant point is that (16) is the correct Galilean equation for *arbitrary* spin in 2+1 space. In other words a Galilean particle of spin S is described by the two-component wave function ψ of Eq. (16) provided that the identification $s = \epsilon(S)$ is made. One thus concludes that the Aharonov-Bohm amplitude is given correctly by Eqs. (14) and (15) in the Galilean limit. Further refinements of this result using fully relativistic wave equations for spin greater than one-half can therefore only yield corrections of order $1/c^2$ to the results obtained here.

The author is indebted to R. E. Marshak for a helpful

discussion. This work is supported in part by U.S. Department of Energy Contract No. DE-AC02-76-ER13065.

Note added.—The crucial role played by the singular states discovered here is apparent in a recent calculation[9] of the second virial coefficient of a system of flux-carrying fermions.

Appendix.—It is shown here that the condition invoked in this paper that H be confined to the surface of a cylinder of radius R is not essential to the principal result. Thus the conclusions of this paper will be seen to apply to any magnetic field which is independent of angle and which vanishes for $r > R$. In particular there is no dependence on the detailed form of H in the limit $R \rightarrow 0$ provided only that

$$\lim_{r \rightarrow 0} \int_0^r H(r')r'dr' = 0 .$$

One begins with the differential equation

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + k^2 + Hs - \frac{m^2}{r^2} - A_\phi^2 + 2mA_\phi \frac{1}{r} \right] \psi = 0,$$

$$r < R ,$$

where

$$H = F(r), \quad A_\phi = \frac{1}{r}\int_0^r r'dr'F(r') .$$

Upon introducing the dimensionless variable $x = r/R$ there results

$$\left[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} + (kR)^2 - \frac{m^2}{x^2} - \frac{1}{x^2}\left(\int_0^x x'dx'f(x') \right)^2 + \frac{2m}{x^2}\int_0^x x'dx'f(x') + sf(x) \right] \psi = 0 ,$$

where $R^2F(r) = f(x)$. Next, one drops terms proportional to $(kR)^2$ and considers separately the two cases (a) $m = 0$ or $s = \epsilon(m)$ and (b) $s = -\epsilon(m)$.

In case (a) one makes the substitution

$$\psi(x) = x^{|m|} \exp\left(-s\int_0^x dx'a(x') \right) \chi(x) ,$$

where $a(x) = RA_\phi(r)$ thereby yielding

$$\left[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} + \frac{2|m|}{x}\frac{d}{dx} - 2sa(x)\frac{d}{dx} \right] \chi = 0 ,$$

which has the regular solution $\chi = 1$. This represents the desired (i.e., regular) solution of the differential equation for $r < R$ and leads to an immediate evaluation of the logarithmic derivative at $r = R$,

$$\psi'(1)/\psi(1) = |m| - s\int_0^1 x\,dx\,f(x)$$
$$= |m| + sa .$$

To complete the calculation in case (a) one now requires continuity with the logarithmic derivative of a solution valid for $r > R$. However, the subsequent analysis is virtually identical to that in the text between Eqs. (7) and (11) and the result is thereby proved.

Case (b) is only slightly more complex. Here one writes

$$\psi = x^{-|m|} \exp\left(-s\int_0^x dx\,a(x') \right) \chi(x)$$

and obtains

$$\left[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} - \frac{2|m|}{x}\frac{d}{dx} - 2sa(x)\frac{d}{dx} \right] \chi(x) = 0 .$$

The solution $\chi = 1$ is in this case not a regular (i.e., admissible) solution of the problem, however. One thus uses the Wronskian to obtain

$$\psi_{\text{reg}}(x) = x^{-|m|} \exp\left(-s\int_0^x dx'a(x') \right) \int_0^x dx'x'^{2|m|-1} \exp\left(2s\int_0^{x'} a(x''-dx'') \right) ,$$

which yields

$$\frac{\psi'_{\text{reg}}(1)}{\psi_{\text{reg}}(1)} = -|m| - s\int_0^1 x\,dx\,f(x) + \frac{\exp\{2s\int_0^1 a(x')dx'\}}{\int_0^1 dx\,x^{2|m|-1}\exp\{2s\int_0^x a(x')dx'\}} .$$

Since the last term in the above is positive, the regular inside solution can be joined to an outside solution $J_{-|m+a|}(kr)$ only if

$$-|m+a| > -|m| + sa .$$

This can be shown to imply

$$\epsilon(m)\epsilon(m+a) > 1 ,$$

which, of course, is never true. Thus irregular solutions are quite generally disallowed for $s = -\epsilon(m)$. Since this is identical to the conclusion obtained in Eq. (11), the proof is complete.

[1]Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[2]J. March-Russell and F. Wilczek, Phys. Rev. Lett. **61**, 2066 (1988).

[3]C. R. Hagen, University of Rochester Report No. UR-1096, 1988 (unpublished).

[4]C. R. Hagen, Phys. Rev. D (to be published).

[5]M. G. Alford and F. Wilczek, Phys. Rev. Lett. **62**, 1071 (1989).

[6]Ph. Gerbert and R. Jackiw, MIT Report No. CTP-1594, 1988 (unpublished).

[7]Ph. Gerbert, Phys. Rev. D **40**, 1346 (1989). This paper introduces a θ parameter to characterize the solutions. In the case that $\theta = \pi/2$ the results become mathematically equivalent to those presented here for the $s = 1$ case, despite the enormous differences between the two approaches.

[8]J. M. Lévy-Leblond, Commun. Math. Phys. **6**, 286 (1967).

[9]T. Blum, C. R. Hagen, and S. Ramaswamy, University of Rochester Report No. UR-1113, 1989 (unpublished).