Scaling Behavior of Localization in Quantum Chaos

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The kicked rotator on a torus is a system with a bounded phase space in which chaotic diffusion occurs for a large enough perturbation strength. The quantum version of this model exhibits localization effects which produce deviations from random-matrix-theory predictions. We show that these localization effects display a scaling behavior which is a counterpart of the scaling theory of one-dimensional Anderson localization in finite samples. We suggest that this behavior can be highly relevant to some general problems of quantum chaos.

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The kicked rotor (KR) was proposed many years ago as a simple model for the investigation of quantum chaos.[1] The most important result obtained up to now is that the quantum KR never reproduces the unbounded diffusion in momentum space that characterizes the classical KR in its chaotic regime. It can, at best, exhibit diffusivelike excitation (in momentum) only on some finite time scale.[2] This "quantum suppression of chaotic diffusion" turned out to be a rather general occurrence in the dynamics of systems subjected to time-periodic perturbations; for example, this very phenomenon lies at the heart of a recently developed theory for the quite different problem of microwave ionization of highly excited hydrogen atoms.[3,4] An important insight on the nature of this suppression was provided by a formal analogy of the quantum KR with a localization problem for a particle on a one-dimensional lattice with a pseudorandom potential;[5] indeed, the quantum limitation of diffusion for the KR and related models has a definite similarity with the Anderson localization of solid-state physics. In order to stress this analogy without overlooking certain conceptual differences the quantum limitation of diffusion is sometimes referred to as "dynamical localization."[6] Thanks to this connection the quantum chaology of periodically driven systems could take advantage of important concepts from solid-state physics.[7] In the present paper we purport to push this analogy one step further by demonstrating the relevance of localization theory to the study of quantum chaos in systems which, unlike the KR, have a *bounded* phase space. As a matter of fact, the classical KR can be considered as a dynamical system on the torus by exploiting its intrinsic periodicity in momentum space. An analogous "folding back on the torus" can be performed for the quantum KR too,[8-10] for particular "resonant" values of the kicking period (see also Ref. 11).

Whereas the original "unbounded" KR could be formally associated with a localization problem on a 1D infinite lattice, the KR "on the torus" corresponds to a localization problem on a finite 1D sample; the Bloch theorem applies, the Bloch number being related to different choices of boundary conditions.

In this paper we shall provide numerical evidence for the validity of a new kind of scaling law for the toral KR. In doing so, an essential step will be to define an appropriate measure of localization in a finite sample.[12] An interesting application of scaling theory to the KR has also been discussed in a recent paper,[13] in which scaling was *assumed* and used to derive a previous estimate for the localization length.[14] (In Ref. 13 it was also assumed that the scaling function is "Ohmic.")

Even though the toral KR discussed here is a time-dependent problem, it poses a situation which is typical of conservative quantum systems with a chaotic classical limit. There, too, a maximal degree of classical chaos is assumed to go along with applicability of random matrix theory (RMT) for modeling the statistics of "chaotic" eigenfunctions and of the spectrum. Nevertheless, usually, intermediate situations exhibit deviations from the ideal RMT case.[15,16] It seems reasonable that the role of localization in determining such deviations, as well as the scaling properties discussed here, can be carried over to this more general setup.

We write the time-dependent Hamiltonian of the classical KR in the form

$$H=\frac{p^2}{2}+k\cos\theta\sum_{n=-\infty}^{+\infty}\delta(t-nT)\,. \tag{1}$$

Notice that (1) defines a dynamical system on a torus as soon as p is taken mod$(2\pi n/T)$, n a fixed integer.

The quantized KR is described by the map

$$\psi(\theta,t+1)=S\psi(\theta,t)=\exp\left\{i\frac{T}{4}\frac{\partial^2}{\partial\theta^2}\right\}\exp(-ik\cos\theta)\times\exp\left\{i\frac{T}{4}\frac{\partial^2}{\partial\theta^2}\right\}\psi(\theta,t)\,, \tag{2}$$

where the ψ function is taken in the middle between two successive kicks[8,9] and t is now the discrete time defined by the number of iterations. We have taken here $\hbar=1$; the classical limit is then defined by $k\to\infty$, $T\to 0$, $kT=$const. When $T=4\pi P/Q$ with P,Q integers, S commutes with translations in momentum by Q. The classical periodicity is then $Q/2P$.

A quantum counterpart for "folding on the torus" is now provided by Bloch's theorem. If the equation for eigenvalues and eigenfunctions of S (the quasienergy eigenvalues and eigenfunctions) is written in the momentum representation,

$$e^{i\lambda}\psi_p^{(\lambda)} = \sum_{q=-\infty}^{+\infty} S_{pq}\psi_q^{(\lambda)}, \tag{3}$$

then, thanks to the periodicity of S, $S_{p+Q,q+Q}=S_{p,q}$, one can put

$$\psi_{kQ+l}^{(\lambda)} = e^{ik\alpha}u_{l,\lambda} \quad (1 \le l \le Q-1), \tag{4}$$

with α the Bloch number chosen in the Brillouin zone $(-\pi/Q, +\pi/Q)$ and $u_{l,\lambda}$ a Q-dimensional vector.[17] For each Bloch number α the matrix $S(\alpha)$ can be considered as a quantization of the classical toral KR. In the language of solid-state physics, it defines the dynamics of a particle in a finite one-dimensional lattice of size Q, and different values of α correspond to different boundary conditions.

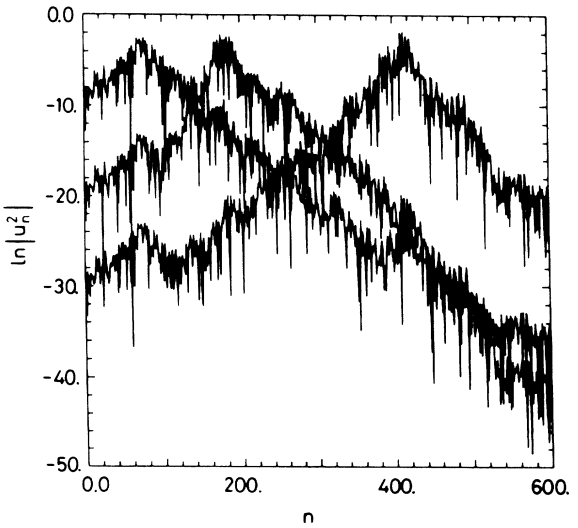

FIG. 1. Three exponentially localized eigenfunctions $\ln|u_n^2|$ vs n of the unitary matrix S_{nm} [Eq. (5)] for $N=600$, $T=0.544$, $k=9.19$, $K=kT=5.00$.

Following Ref. 8 we choose α in the band center, i.e., we choose periodic boundary conditions. Moreover, one can restrict the study of S to the invariant subspace of odd states. Then the matrix elements of S are

$$S_{nm} = \frac{e^{i(T/4)(n^2+m^2)}}{2N+1}\sum_{l=1}^{2N+1}\left\{\left[\cos\left(\frac{2\pi(n-m)l}{2N+1}\right)-\cos\left(\frac{2\pi(n+m)l}{2N+1}\right)\right]e^{-ik\cos[2\pi l/(2N+1)]}\right\}, \tag{5}$$

where $Q=2N+1$ and $n,m=1,2,\ldots,N$.

The dynamical properties of the classical KR depend on the parameter $K=kT$. For $K\gg 1$ one has fully developed chaos *on the torus*. According to generally accepted views, one should expect the spectral properties of the quantum toral KR in the semiclassical region to be well described by RMT. It turned out that fully developed chaos ($K\gg 1$) is not a sufficient condition for RMT to apply. Indeed, in such a situation one can have an almost pure Poissonian distribution. Such deviations from the RMT occur because of missing or incomplete delocalization of eigenfunctions. Indeed, when N—the size of the torus or of the sample—is large enough, a wave packet initially concentrated on some site n_0 dynamically evolves in the beginning just as it would for $N=\infty$, because it is so narrow that it does not "feel" the boundary conditions. However, the spreading of the wave packet will be stopped by interference after some finite time and "frozen" in a (roughly) exponential configuration with the probability at site n decaying like $\exp(-2l_s^{-1}|n-n_0|)$. The localization length is[2] $l_s \approx k^2/2$. In this case the eigenfunctions would be exponentially localized with a localization length[14] being half as large as l_s:

$$l \approx l_s/2 \approx k^2/4. \tag{6}$$

[We stress here that a condition for the validity of (6) is that $K\gg 1$, $k\gg 1$. The localization length cannot be less than k, which approximately gives the number of states coupled by one kick.] See, for example, Fig. 1 which shows three exponentially localized odd eigenfunctions (only half of each eigenfunction is shown) for the toral KR with $N=600$, $K=5$, and $k=9.2$.

The above discussion suggests that the effectiveness of localization should depend on the localization variable k^2/N. In solid-state physics localization in finite samples is closely related to the residual conductance of the samples themselves.[18,19] The assumption that conductance depends only on the ratio between the size of the sample and the localization length for the infinite sample is the core of the scaling theory of localization. There are serious difficulties in finding a KR equivalent for the solid-state scaling assumption if the latter is formulated in terms of conductance. In order to meaningfully define such physical concepts as conductance, mean free path, and so on, within the mathematical formalism of Anderson localization, an elaborate theoretical apparatus is needed and great care must be taken to carry this apparatus across the still precarious bridge which connects the KR with tight-binding models of solid state.

Fortunately enough a formulation of scaling theory exists that under suitable modifications can be applied to the KR. According to Ref. 19 the scaling *Ansatz* is equivalent to postulating the existence of a function $f(x)$ such that

$$\xi_N/\xi_\infty = f(x), \quad x = N/\xi_\infty, \tag{7}$$

where ξ_∞ is the localization length for an infinite sample.

Here ξ_N is defined by means of the transfer-matrix formalism. It is a characteristic length of the exponential decay of the eigenfunctions in the finite sample of length N and $\xi_N \to \xi_\infty$ for $N \to \infty$. In the approach of Ref. 19 the quantity ξ_N is directly related to the residual conductance via Landauer's formula. In order to apply (7) to the toral KR, we need to define an equivalent for ξ_N, but such a definition cannot be taken *verbatim* from the tight-binding model underlying (7). In the first place, a transfer-matrix formalism for the KR is meaningless when $k \sim N$ because then the rank of the transfer matrix (which is given by the number of sites effectively coupled by one kick) is comparable to the size of the sample. Moreover, in truly intermediate situations there is no practical way to read something like a scale of exponential decay off actual eigenfunctions, which typically look like the one displayed in Fig. 2.

We need, therefore, to define a parameter measuring the localization of eigenfunctions even in cases when localization is not obviously exponential. Such a parameter was introduced in Ref. 12 via the Shannon entropy of eigenfunctions. For a normalized eigenfunction assuming a value u_n at site n the Shannon entropy is defined by

$$H_N(u) = -\sum_{n=1}^{N} |u_n^2| \ln |u_n^2| . \tag{8}$$

It is seen that $H_N(u)$ is essentially the logarithm of the number of sites significantly populated by the given eigenfunction u. If all sites were equally populated, then $H_N(u)$ would be $\ln N$.

However, in the case of extreme delocalization, each eigenstate looks like a typical eigenvector of a random matrix. The average entropy $\bar{H}$ of such an eigenvector can be explicitly computed, because any eigenvector of a random matrix is a random vector, uniformly distributed over the surface of an N-dimensional unit sphere. For large N, the probability distribution for any component of such a "microcanonical" random vector approaches a Gaussian distribution. Then one obtains[12] $\bar{H}_N = \psi(N/2+1) - \psi(\frac{1}{2})$, where $\psi(z)$ is the logarithmic derivative of the gamma function. Therefore, for $N \gg 1$, $\bar{H}_N = \ln(\alpha N/2) + 1/N + O(1/N^2)$ with $\alpha \approx 0.96$.

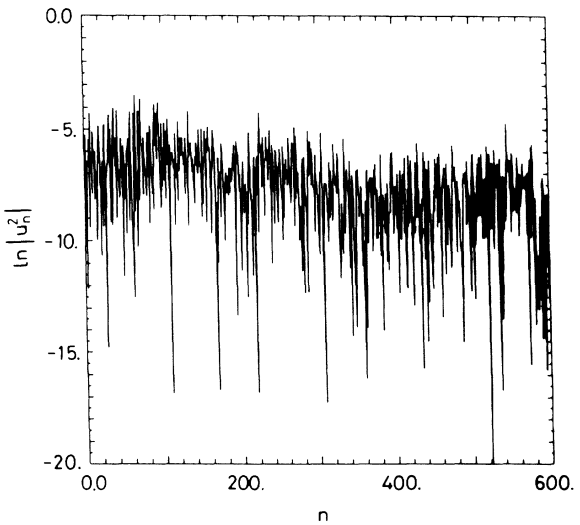

FIG. 2. Typical delocalized eigenfunction of the unitary matrix S_{nm} [Eq. (5)] for $N=600$, $T=0.146$, $k=34.1$, $K=kT=4.98$.

In other words, should we define the effective number of states spanned by the eigenfunction as $d(u) = \exp[H_N(u)]$, then $d(u)$ in the delocalized case would be approximately equal to $N/2$. For this reason we shall instead define the effective dimension for one eigenfunction by

$$d_N(u) = N \exp[H_N(u) - \bar{H}_N] = (2/\alpha)\exp H_N(u) . \tag{9}$$

With this definition, in the opposite case of exponential localization (when $1 \ll l \ll N$) we find $H_N(u) = 1 + O(1/l)$ and therefore $d = d_\infty = 2le/\alpha + o(1) \approx 2.8 l_s$.

A measure of the localization of eigenfunctions similar to Eq. (9) has been used in different quantum problems in Refs. 20 and 21, without the normalization factor $2/\alpha$. This factor is dictated by the random nature of chaotic eigenfunctions and is essential for our present purposes.

Using d_N as a measure of the delocalization of eigenfunctions in the finite sample, we can formulate a scaling assumption for the KR in the spirit of Eq. (7). Recalling that for the KR ξ_∞ is proportional to k^2, we conjecture a scaling law of the form

$$\frac{d}{N} = g\left(\frac{k^2}{N}\right), \quad d = \langle d_N \rangle . \tag{10}$$

Angular brackets in (10) denote the average over all eigenfunctions of a given S matrix. Indeed, it is known that the validity of a universal scaling behavior is restricted by macroscopic quantum effects which take the form of huge fluctuations and need somehow to be washed out if a single scaling law is to be exposed. Of course, these fluctuations convey important physical information that should be investigated, even in the KR case.

We tested the scaling (10) for our model (5) by numerically computing the eigenfunctions of the matrix (5) for widely different values of N and k in the ranges $200 \le N \le 860$ and $1 \le k \le 239$. The classical stochasticity parameter was taken in all cases to be $K=5$, which corresponds to negligible residual islands of stability. Even though the fluctuations of d_N of individual eigenfunctions around the averaged values are rather large, there is excellent evidence of scaling in the mean (Fig. 3). The numerical results show a linear dependence of $d = \langle d_N \rangle$ for small k^2/N in accordance with the theoretical predictions (6) for exponentially localized eigenfunctions. As $k^2/N \to \infty$, the curve approaches the saturation value 1. Its increase is very slow and according to our data it is consistent with a law $\beta \simeq 1 - 0.53(k^2/N)^{-1/2}$ which actually fits the numerical data

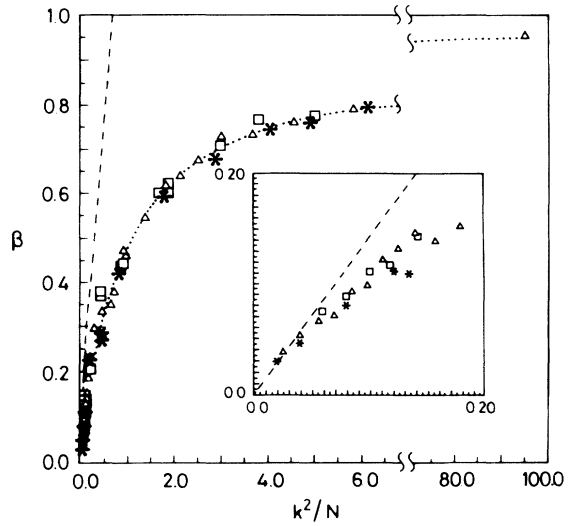

FIG. 3. Delocalization $\beta = d/N$ vs localization variable k^2/N for $N \approx 400$ (squares), $N \approx 600$ (triangles), and $N \approx 800$ (stars). Inset: Magnification of the region of small localization variable. The dashed lines are calculated for exponentially localized eigenstates using the theoretical expression for d_∞. The dotted line is the dependence $\beta \simeq 1 - 0.53(k^2/N)^{-1/2}$.

for $k^2/N \geq 1$.

Our numerical results demonstrate that localization effects in the toral KR exhibit a scaling behavior which is a natural counterpart for the scaling theory of one-dimensional Anderson localization. Of course, this scaling law calls for further investigation. For example, it is known[22] that the simple estimate (6) fails when the period $T > 1$, for in that case l_s depends not only on k but also on T. Since (6) determines the scaling curve for $k^2/N \ll 1$, a similar breakdown of the scaling behavior is expected in the essentially quantum region $T > 1$. Nevertheless, the scaling property should be expected to be still valid in the proper scaling variable ξ_∞/N. We therefore expect the scaling to be restored on changing variables from k^2/N to $2D/N$, with D the quantum diffusion coefficient measuring the initial rate of spreading of a wave packet; indeed, the theoretical argument leading to (6) entails the proportionality of D and ξ_∞.

Though preliminary, our results yield nevertheless remarkable indications. A first one is that, since deviations from RMT statistics in the region of full classical chaos are just due to localization, these deviations must exhibit a scaling behavior themselves. For example, it has been shown[12] that the degree of repulsion of quasienergy levels is directly related to the above defined spectral parameter d/N; therefore, the degree of repulsion must display the same scaling as that parameter, and this is indeed confirmed by some preliminary numerical results of ours.

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