PHYSICAL REVIEW LETTERS

VOLUME 64 | 29 JANUARY 1990 | NUMBER 5

Structure of Temperley-Lieb Algebras and Its Application to 2D Statistical Models

Dan Levy

Physikaliches Institut, University of Bonn, Nussallee 12, D-5300 Bonn 1, West Germany

(Received 1 November 1989)

A simple derivation of the ideals in the Temperley-Lieb algebras is presented together with a convenient basis. Using this basis it is shown how to derive the eigenvalue equations for operators defined on the algebra and a relation to the Bethe-*Ansatz* equations is noticed. The irreducible content of the spin-$\frac{1}{2}$ XXZ model is also discussed.

PACS numbers: 05.50.+q, 02.20.+b

The Temperley-Lieb (TL) algebras were discovered by Temperley and Lieb[1] who used them to relate the spectrum of the six-vertex model to that of the Potts models (the latter include the Ising model). Their relevance to 2D integrable statistical-mechanics models was further clarified by Baxter.[2] He showed how they arise from solutions to the integrability conditions (the Young-Baxter equations). Nowadays they are also of interest due to the fact that any representation of them is automatically a representation of a corresponding braid group. Braid groups may have yet a broader range of physical applications. The TL algebra A_N is an associative algebra (i.e., equipped with an involution operation later denoted by a dagger) over the complex numbers that is generated by the unit element 1 and the letters $e_1, \ldots, e_N$ satisfying

$$e_i^2 = \sqrt{q}\, e_i, \quad q \geq 0, \tag{1a}$$

$$e_i e_{i\pm 1} e_i = e_i, \tag{1b}$$

$$[e_i, e_j] = 0, \quad \forall\, |i-j| > 1. \tag{1c}$$

The first detailed account of their mathematical structure and its dependence on the value of the parameter q was given by Jones.[3]

In a typical physical application, the physical system will be defined by an operator $O(A_N)$ which we wish to diagonalize and a particular representation of A_N (we assume free boundary conditions). Baxter[2] had noted that the partition function can, in principle, be computed purely algebraically just by using (1). Here we extend this idea to any operator $O(A_N)$ and turn it into a practical program, at least for several cases of interest. Our basic result is a useful basis for the primitive left ideals of A_N. Once it is established which ones are realized in the particular representation defining the model, we can derive the eigenvalue equations separately on each, and thus break up the problem to more manageable pieces.

The paper is organized as follows: We begin by using elementary means to find the primitive left ideals and construct the above-mentioned basis. Then we show how it can be used to obtain the eigenvalue equations. We end up with a discussion of the irreducible content of the XXZ model.

Primitive left ideals of A_N.—We recall some basic definitions. A primitive left ideal is a collection of elements $I^L(A_N)$ satisfying $a \cdot i \in I^L(A_N)\ \forall\ a \in A_N$ and $i \in I^L(A_N)$ which contains no such smaller subcollection. There is a natural relation between primitive left ideals and irreducible representations.[4] Next, we call any product of letters a word (1 is an empty word). Following Jones, a word is called reduced if it cannot be made any shorter using (1). All reduced words which are equal under (1c) will be considered equivalent. Any word is proportional to a reduced word, e.g., $e_1e_3e_2e_1e_3 = e_1e_3e_2e_3e_1 = e_1e_3e_1 = e_1e_1e_3 = \sqrt{q}\, e_3e_1 = \sqrt{q}\, e_1e_3$, and therefore the set of nonequivalent reduced words spans the algebra. Their number N_R (the empty word included) is given[3] by the Catalan number

$$\frac{1}{(N+2)}\binom{2(N+1)}{N+1}.$$

We will start by assuming that they are linearly independent and therefore can provide a basis (this is termed the generic case). In this case $\dim(A_N)=N_R$.

To get the ideals we introduce the following new concept. We say that a jump occurs in the reduced word w whenever an index on two neighboring letters differs by more than 1. We define $N(w)$ to be the number of such occurrences in a given word w (not necessarily reduced), and $N(\sim w)$ to be the minimal value of $N(w)$ in the equivalence class of the reduced form of w [$N(e_i)\equiv 0$ and $N(1)\equiv -1$]. The usefulness of $N(\sim w)$ lies in the fact that it cannot decrease upon multiplication,

$$N(\sim w_1 w_2)\geq \max[N(\sim w_1),N(\sim w_2)]\,, \tag{2}$$

and in the fact that it takes integer values in the limited range from -1 to $[(N-1)/2]$. Define I_k to contain all words w such that $N(\sim w)\geq k$. Then the I_k's form a family of nested two-sided ideals ($I_k \subset I_{k-1}$), I_{-1} being the whole of A_N.

Consider the following set of reduced words given by $S_k \equiv e_1 e_3 \cdots e_{2k+1}$ ($S_{-1}\equiv 1$). S_k has k jumps and any word with k jumps is proportional to a (nonreduced) words of the form $w_L S_k w_R$, with $w_L S_k$ and $S_k w_R$ reduced. Out of the I_k's, only the last one with $k=[(N-1)/2]$ is primitive. Indeed,

$$S_k w S_k \propto S_k, \quad \forall\, w \in A_{2k+2}\,. \tag{3}$$

So only $S_{[(N-1)/2]}$ satisfies the necessary and sufficient condition to generate a primitive ideal in A_N.[4] Can one construct primitive ideals in A_N based on the other S_k's? S_k is a barrier for $e_1,\dots,e_{2k+2}$. If one can find $Z_k^N(e_{2k+3},\dots,e_N)$ satisfying $e_i Z_k^N=0 \ \ \forall \ \ i\geq 2k+3$, then the collection of elements of the form $(wS_k)Z_k^N$ is a primitive left ideal where the reduced words wS_k have exactly k jumps. For many practical purposes, once we know Z_k^N exists we can forget about it and simply define $w'(wS_k)$ to be zero if the result contains more than k jumps.

When $e_i^\dagger = e_i$ one can show that a nontrivial Z_k^N must be (up to an overall factor) of the form $Z_k^N = 1-E_k^N$, where E_k^N is a linear combination of nonempty reduced words, acting as a unique two-sided identity on the letters $e_{2k+3},\dots,e_N$. One can obtain[5] an inductive formula for E_k^N, depending on the number of letters $N-(2k+2)$ and q, which tells the following: For $q\geq 4$, E_k^N exists for all allowed k's for all N. Hence each A_N has a full set of $[(N+3)/2]$ primitive ideals. In this case the reduced words are indeed linearly independent. We can now give the basis announced earlier. Defining $C_{n_1}^{n_2} \equiv e_{n_2} e_{n_2-1}\cdots e_{n_1}$ ($n_2 \geq n_1$) then the basis vectors are given by

$$^1C_M \equiv {}^1C_{m_1,\dots,m_{k+1}} \equiv C_1^{m_1} C_3^{m_2}\cdots C_{2k+1}^{m_{k+1}}, \\ 1\leq m_1 < \cdots < m_{k+1} \leq N, \quad m_i \geq 2i-1\,. \tag{4}$$

The inequality imposed on the m_i's guarantees that the ^1C_M's are reduced. Their number for a given k is

$$\binom{N}{k+1}-\binom{N}{k-1}$$

[where $\binom{N}{i}=0$ for $i<0$], and this gives the dimension of the generic I_k^L.

For $q=4\cos^2(\pi/p)$, where p is an integer (for $q<4$ these are the only q values for which $e_i^\dagger = e_i$ is consistent; see Jones[3]), a nontrivial Z_k^N exists only for $k > \max\{-2,[(N-p)/2]\}$. As a result the number of primitive ideals cannot grow (with N) beyond $[(p+1)/2]$. Moreover, $Z_{-1}^{(p-2)}=0$; that is, linear relations appear between reduced words. The ^1C_M's still span the surviving ideals but as explained, they are linearly dependent. We will return to this point shortly.

The reduced k-jump words in $I_k^L(A_N)$ divide into words which contain e_N and words which do not. The latter are words on A_{n-1}. The former must have the e_N letter on the rightmost decreasing chain and therefore can be factored as $^1C_{m_1,\dots,m_k} C_{2k+1}^N$ where the left factor is a $(k-1)$-jump word on A_{N-1}. We can summarize this by the following symbolic formula:

$$I_k^L(A_N) = I_k^L(A_{N-1}) + I_{k-1}^L(A_{N-1})C_{2k+1}^N\,. \tag{5}$$

Equation (5), which shows how to construct a basis for the ideals of A_N out of the basis for the ideals of A_{N-1}, is equivalent in its contents to a Bratteli diagram[3] (one has to attach a primitive left ideal to each node). For $e_i^\dagger = e_i$, $q\geq 4$, it is identical with the basis given in (4). For $q=4\cos^2(\pi/p)$, Eq. (5) tells us how to construct a linearly independent basis of reduced words for the surviving ideals. One just has to ignore any I_{k-1}^L for which Z_{k-1}^L does not exist.

Eigenvalue equations.—The eigenvectors of $O(A_N)$ will be represented in the vector space A_N in each k-jump sector by linear combinations of the type $\sum_M a_M\, {}^1C_M$. Since the most general $O(A_N)$ is a linear combination of reduced words, it is enough to know the action of single letters upon the basis. This can be evaluated explicitly using (1) and will be given for general k elsewhere. Here we give an illustrative example. As $O(A_N)$ we take the Hamiltonian of a Potts or XXZ chain at criticality with free boundary conditions,

$$H=\sum_{i=1}^N e_i\,. \tag{6}$$

Suppose that the zero-jump ideal exists. Then the eigenvalue equation for it reads

$$H\sum_{m=1}^N a_m\, {}^1C_m = E\sum_{m=1}^N a_m\, {}^1C_m\,. \tag{7}$$

In order to evaluate the left-hand side we need the following result:

$$e_i\,{}^1C_m=(\delta_{i,m-1}+\sqrt{q}\,\delta_{i,m}+\delta_{i,m+1})\,{}^1C_i\,. \qquad (8)$$

Inserting (8) into (7), performing the sum over m on the left-hand side, and collecting coefficients of each ${}^1C_i$ gives

$$a_{i-1}+(\sqrt{q}-E)a_i+a_{i+1}=0,\quad 1\le i\le N\,, \qquad (9)$$

where the boundary conditions are encoded in $a_0=a_{N+1}=0$. Equation (9) is identical in form to the $(n=1)$-sector Bethe-*Ansatz* equation of the XXZ model.[6] The general k-jump-sector equation is similar (but not identical) to the $(k+1)$-sector XXZ equation. The reason for this will become clear when we discuss the decomposition of the XXZ representation.

How can we formulate the eigenvalue problem when the reduced words are not linearly independent? As explained, we can isolate via Eq. (5) a linearly independent set, but then, in addition to (1), we will also need the condition $Z_{-1}^{(p-2)}=0$ (which hides a fairly complicated relation whose complexity grows rapidly with p) in order to calculate the action of e_i upon this basis. Instead, we propose to keep using ${}^1C_M$ together with (1), ignoring the extra relation. This will introduce many false eigenvectors which are in fact zero vectors expressed as nontrivial linear combinations of ${}^1C_M$'s. However, because of the rather trivial fact that (anything)$\cdot 0=0$, they have to form invariant subspaces within the surviving k-jump sector. We further claim that these invariant subspaces will be l-jump spaces with $l<k$. We are therefore looking for a vector V satisfying

$$\begin{aligned} &e_iV=0,\quad \forall\, i\ge 2l+3,\\ &V=\sum_M a_M\,{}^1C_{1,3,\ldots,2l+1,m_{l+2},\ldots,m_{k+1}}\,. \end{aligned} \qquad (10)$$

If nontrivial a_M's solving (10) exist, one can generate an l-jump subspace from V by multiplying it from left with ${}^1C_{m_1,\ldots,m_{l+1}}$. Hence, out of the H eigenvectors so obtained all those of the form $\sum_M a_M\,{}^1C_{m_1,\ldots,m_{l+1}}V$ have to be rejected for all pairs (l,V) which solve (10). Omitting here the details, it turns out that a nontrivial solution to (10), if it exists, is unique for a specific l value. The allowed l's are determined by the simple condition

$$l=N-pm-k,\quad m\ \text{integer}\ge 1,\quad -1\le l<k\,. \qquad (11)$$

To illustrate the above points we take as an example $N=3$. The generic algebra has three sectors with $k=-1,0,1$ whose dimensions are respectively 1,3,2. In the Ising case ($q=2\Longrightarrow p=4$), the -1 sector does not exist and the 0- and 1-jump sectors are each of dimension 2. So if the generic basis is used for the zero-jump Ising sector, one spurious vector is introduced. Indeed, Eq. (11) gives $l=-1$ for $N=3$, $p=4$, $k=0$, and $m=1$. The -1-jump subspace is one dimensional (irrespective of N). This spurious vector is annihilated by all the e_i's, so from (6) it is clear that it has a zero eigenvalue under H. In general, if we have the solutions of the generic eigenvalue equations up to k, we can also construct the k-sector spectrum for the special q values. In relation to this, using the solutions of (11), it should be possible to obtain the true dimension of a surviving I_k^L, for general N,k,p, by subtracting spurious vectors from the generic I_k^L. One should pay attention to the fact that the zero vectors comprising the l-jump subspace of the k sector may themselves be linearly dependent. This can be decided by examining the subspaces of the l-jump subspace itself [in (11) one looks for $l^{(2)}$'s which solve the equation with $k=l$ on the right-hand side]. If an l-jump subspace has yet such a smaller $l^{(2)}$-jump space which does not show in the original k space, then the vectors in the l subspace are linearly dependent. One has to iterate this procedure until no new invariant subspaces are found. If one defines inductively

$$\{l^{(i)}\}=N-pm-\{l^{(i-l)}\}\,, \qquad (12)$$

where $\{l^{(i)}\}$ is the family of solutions obtained in the ith step, with any $l^{(i)}$ value already obtained in a previous step discarded and $\{l^{(0)}\}-k$, then we have

$$\dim I_k^L[\text{at } q=4\cos^2(\pi/p)]=\sum_i(-1)^i\sum_{l^{(i)}}\dim I_{l^{(i)}}^L(\text{generic})\,. \qquad (13)$$

Content of the spin-$\frac{1}{2}$ XXZ model.—The following discussion[7] will serve to show how the I_k^L's are realized in a concrete representation, to learn what can happen when $e_i\ne e_i^\dagger$,[8] and to clarify some aspects of the Bethe-*Ansatz* solution. In the XXZ representation of the TL algebra, the e_i's are represented by the following matrices:

$$V_l(\alpha)\equiv\exp(i\alpha\sigma_l^z)P_l\exp(i\alpha\sigma_l^z),\quad \sqrt{q}=2\cos(2\alpha)\,, \qquad (14a)$$

$$P_l\equiv\tfrac14(\sigma_l^+\sigma_{l+1}^-+\sigma_l^-\sigma_{l+1}^+)+\tfrac12(1-\sigma_l^z\sigma_{l+1}^z)\,, \qquad (14b)$$

where $\sigma_l^i=I\otimes\cdots\otimes I\otimes\sigma^i\otimes I\otimes\cdots\otimes I$, I is a 2×2 unit matrix, and $\sigma^i(i=z,+,-)$ is a 2×2 Pauli matrix standing in the lth position of the $(N+1)$-fold tensor product.

Since $P_l^\dagger=P_l$ we have $V_l^\dagger(\alpha)=V_l(\alpha)$ only if α is pure imaginary, which corresponds to $q\ge4$. For real α, the above provides a nonunitary representation with $q<4$. The V_l's act on a chain of $N+1$ spins, where each spin can be either up ($|\uparrow\rangle$) or down ($|\downarrow\rangle$). The nontrivial part of the action is on the $l,l+1$ subspace:

$$\begin{aligned} &V_l\,|\uparrow\uparrow\rangle=V_l\,|\downarrow\downarrow\rangle=0;\\ &V_l\,|\uparrow\downarrow\rangle=\exp(2i\alpha)V_l\,|\downarrow\uparrow\rangle=\exp(2i\alpha)\,|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\,. \end{aligned} \qquad (15)$$

We see that the V_l's conserve the number of up (and down) spins, so we can work in a subspace of fixed $n\le[N/2]$ up spins. Next, we ask which jump sectors can be accommodated in such a subspace. If I_k^L is contained, we must find a spin state which is not annihilated by S_k. From (15) and the definition of S_k it is clear that the spins at positions $i,i+1$ must be relatively flipped for $i=1,3,\ldots,2k+1$, in order to get a nonzero result. It follows that in an n-up-spin subspace, the highest jump sector we can find is I^L_{n-1}.

We return to our previous example of the A_3 algebra in order to explain how the construction proceeds and later state the general result. We have in total four spins, and we take the subspace $n=1$. We can find a unique state which is nonzero only under the action of V_1,

$$|\,\rangle_0 \equiv |\uparrow\downarrow\downarrow\downarrow\rangle . \qquad (16)$$

Then $I^L_0(A_3)$ is realized by the states $V_1|\,\rangle_0$, $V_2V_1|\,\rangle_0$, $V_3V_2V_1|\,\rangle_0$. Using (15) it is easy to see that these states are nonzero and linearly independent. Because of the uniqueness of $|\,\rangle_0$ there can be only one copy of a zero-jump sector, so we can only look now for a -1-jump sector. This means that we are looking for a state which is annihilated by all the V_i's. We can get such a state by "α antisymmetrization" of $|\,\rangle_0$, namely,

$$|\,\rangle_{-1} = |\uparrow\downarrow\downarrow\downarrow\rangle - \exp(2i\alpha)|\downarrow\uparrow\downarrow\downarrow\rangle + \exp(4i\alpha)|\downarrow\downarrow\uparrow\downarrow\rangle - \exp(6i\alpha)|\downarrow\downarrow\downarrow\uparrow\rangle \qquad (17)$$

is annihilated by all V_i's.

The result for the general case is as follows: In an n-up-spin subspace, one can construct [explicit generalizations of (16) and (17) will be given elsewhere] representations for all generic I^L_k's with $k=-1,0,\ldots,n-1$, one copy for each. For $\alpha \neq p'\pi/p$ (p',p integers) the n-up-spin subspace breaks into a direct sum of these I^L_k's. For $\alpha = p'\pi/p$ the construction of the I^L_k's remains the same but linear dependences can appear. Returning to the example one finds for $\alpha=\pi/8$ (the Ising value)

$$|\,\rangle_{-1} = \left\{ \exp\left(\frac{-i\pi}{4}\right) V_1 - \left[1+\exp\left(\frac{-i\pi}{2}\right)\right] V_2V_1 - \exp\left(\frac{i3\pi}{4}\right) V_3V_2V_1 \right\} |\,\rangle_0 , \qquad (18)$$

i.e., $I^L_{-1}(A_3) \subset I^L_0(A_3)$ for this value of α. This also follows from (11). However, the difference with the previous discussion is that $|\,\rangle_{-1}$ is not a zero vector. It appears as a "true" subspace of $I^L_0(A_3)$. This is possible due to the fact that the representation is not unitary. In this case, the 4D 1-up-spin subspace does not decompose. It has a 3D invariant subspace containing further a one-dimensional invariant subspace, but the complementary subspaces are not invariant.[9] It is amusing to note that for the purpose of identifying the spectrum of the unitary generalized $[q=4\cos^2(\pi/p)]$ Potts models within the spectrum of the nonunitary XXZ model with the same value of q, the discussion following Eq. (11) carries over. The solutions we discard may come from true eigenvectors of the XXZ Hamiltonian, but they are not "true" k-jump states and therefore cannot contribute to the Potts spectrum. It is plausible that the Bethe-*Ansatz* solution of Ref. 6 "sees" in each n-up-spin subspace just the $(n-1)$-jump sector it contains. I hope to elaborate on this as well as on other topics elsewhere.

I am indebted to Anton Bovier who collaborated with me in early stages of this work and contributed to the ideas presented here. I would also like to acknowledge financial support from the Deutsche Forschungsgemeinschaft.

[1]H. N. V. Temperley and E. H. Lieb, Proc. Roy. Soc. London, A **322**, 25 (1971).

[2]R. J. Baxter, J. Stat. Phys. **28**, 1 (1982); *Exactly Solved Models in Statistical Mechanics* (Academic, New York, 1982), Chap. 12.

[3]V. F. R. Jones, Invent. Math. **72**, 1 (1983). See also V. F. R. Jones, Ann. Math. **126**, 335 (1987); H. Wenzl, Invent. Math. **92**, 349 (1988); P. P. Martin, J. Phys. A **21**, 577 (1988).

[4]M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley, Reading, MA, 1964); Wu-Ki Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).

[5]This formula is quoted in several places. See V. F. R. Jones, Notices Amer. Math. Soc. **1986**, 219 ; and Martin (Ref. 3) who quotes it from an unpublished work by Temperley.

[6]F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel, J. Phys. A **20**, 6397 (1987).

[7]For a different approach, using the theory of quantum groups, to the study of the same model, see V. Pasquier and H. Saleur, Saclay Report No. Sph-T-89-031, 1989 (to be published).

[8]It can be interesting, at least from an algebraic point of view, to look at the structure of the algebra when both e_i and $e_i^\dagger$ are included.

[9]For the explicit form of the matrices see Martin, Ref. 3.